

RESOLUTION OF EXTENSIONS OF PICARD 2-STACKS

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ABSTRACT. Let \mathbf{S} be a site. First we define the 3-category of torsors under a Picard \mathbf{S} -2-stack and we furnish

- (1) a parametrization of the equivalence classes of objects, 1-arrows, 2-arrows and 3-arrows of the 3-category of torsors under a Picard \mathbf{S} -2-stack by the cohomology groups $H^i(\mathrm{R}\Gamma(-))$ of the derived functor of the functor of global sections, and
- (2) a geometrical description of the cohomology groups $H^i(\mathrm{R}\Gamma(-))$ applied to length 3 complexes of abelian sheaves via torsors under a Picard \mathbf{S} -2-stack.

Then we describe extensions of Picard \mathbf{S} -2-stacks in term of torsors under a Picard \mathbf{S} -2-stack which are endowed with a group law on the fibers. As a consequence of such a description we get an explicit right resolution of the 3-category of extensions of Picard \mathbf{S} -2-stacks in terms of 3-categories of torsors under a Picard \mathbf{S} -2-stack. Using the dictionary between the derived category $\mathcal{D}(\mathbf{S})$ of abelian sheaves on \mathbf{S} and the category of Picard \mathbf{S} -2-stacks, we rewrite this categorical right resolution in homological terms.

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INTRODUCTION

Let \mathbf{S} be a site. A *gr- \mathbf{S} -2-stack* $\mathbb{G} = (\mathbb{G}, \otimes, \mathbf{a}, \pi)$ is an \mathbf{S} -2-stack in 2-groupoids \mathbb{G} equipped with a morphism of \mathbf{S} -2-stacks $\otimes : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}$, called the group law of \mathbb{G} , with a natural 2-transformation of \mathbf{S} -2-stacks \mathbf{a} , called the *associativity*, which expresses the associativity constraint of the group law \otimes of \mathbb{G} , and with a modification of \mathbf{S} -2-stacks π which expresses the obstruction to the coherence of the associativity \mathbf{a} (i.e. the obstruction to the pentagonal axiom) and which satisfies the coherence axiom of Stasheff's polytope (see (1.5) or [4§4] for more details). Moreover we require that for any object X of $\mathbb{G}(U)$ with U an object of \mathbf{S} , the morphisms of \mathbf{S} -2-stacks $X \otimes - : \mathbb{G} \rightarrow \mathbb{G}$ and $- \otimes X : \mathbb{G} \rightarrow \mathbb{G}$, called respectively the left and the right multiplications by X , are equivalences of \mathbf{S} -2-stack.

A *strict Picard \mathbf{S} -2-stack* (just called Picard \mathbf{S} -2-stack) $\mathbb{P} = (\mathbb{P}, \otimes, \mathbf{a}, \pi, \mathbf{c}, \zeta, \mathbf{h}_1, \mathbf{h}_2, \eta)$ is a gr- \mathbf{S} -2-stack $(\mathbb{P}, \otimes, \mathbf{a}, \pi)$ equipped with a natural 2-transformation of \mathbf{S} -2-stacks \mathbf{c} , called

1991 *Mathematics Subject Classification.* 18G15, 18D05.

Key words and phrases. Picard 2-stacks, torsors, extensions, resolution.

the *braiding*, which expresses the commutativity constraint for the group law \otimes of \mathbb{P} , with a modification of \mathbf{S} -2-stacks ζ which expresses the obstruction to the coherence of the braiding c , with two modifications of \mathbf{S} -2-stacks $\mathfrak{h}_1, \mathfrak{h}_2$ which express the obstruction to the compatibility between a and c (i.e. the obstruction to the hexagonal axiom), and finally with a modification of \mathbf{S} -2-stacks η which expresses the obstruction to the strictness of the braiding c . We require also that the modifications $\zeta, \mathfrak{h}_1, \mathfrak{h}_2$ and η satisfy some compatibility conditions. Picard 2-stacks form a 3-category $2\mathcal{P}\text{ICARD}(\mathbf{S})$ whose hom-2-groupoid consists of additive 2-functors, morphisms of additive 2-functors and modifications of morphisms of additive 2-functors.

As Picard \mathbf{S} -stacks are the categorical analogue of length 2 complexes of abelian sheaves over \mathbf{S} , the concept of Picard \mathbf{S} -2-stacks is the categorical analogue of length 3 complexes of abelian sheaves over \mathbf{S} . In fact in [12], it is proven the existence of an equivalence of categories

$$(0.1) \quad 2\text{st}^{bb} : \mathcal{D}^{[-2,0]}(\mathbf{S}) \longrightarrow 2\mathcal{P}\text{ICARD}^{bb}(\mathbf{S})$$

where $\mathcal{D}^{[-2,0]}(\mathbf{S})$ is the full subcategory of the derived category $\mathcal{D}(\mathbf{S})$ of complexes of abelian sheaves over \mathbf{S} such that $H^{-i}(A) \neq 0$ for $i = 0, 1, 2$, and $2\mathcal{P}\text{ICARD}^{bb}(\mathbf{S})$ is the category of Picard 2-stacks obtained from the 3-category $2\mathcal{P}\text{ICARD}(\mathbf{S})$ by taking as objects the Picard 2-stacks and as arrows the equivalence classes of additive 2-functors. We denote by $[]^{bb}$ the inverse equivalence of 2st^{bb} .

Let \mathbb{G} be a gr- \mathbf{S} -2-stack. A *right \mathbb{G} -torsor* $\mathbb{P} = (\mathbb{P}, m, \mu, \Theta)$ is an \mathbf{S} -2-stack in 2-groupoids \mathbb{P} equipped with a morphism of \mathbf{S} -2-stacks $m : \mathbb{P} \times \mathbb{G} \rightarrow \mathbb{P}$, called the action of \mathbb{G} on \mathbb{P} , with a natural 2-transformation of \mathbf{S} -2-stacks μ which expresses the compatibility between the action m and the group law of \mathbb{G} , with a modification of \mathbf{S} -2-stacks Θ which expresses the obstruction to the compatibility between μ and the associativity a underlying \mathbb{G} (i.e. the obstruction to the pentagonal axiom) and which satisfies the coherence axiom of Stasheff's polytope. Moreover we require that \mathbb{P} is locally equivalent to \mathbb{G} and also that \mathbb{P} is locally not empty. If \mathbb{G} acts on the left side, we get the notion of left \mathbb{G} -torsor. A *\mathbb{G} -torsor* $\mathbb{P} = (\mathbb{P}, m^l, m^r, \mu^l, \mu^r, \Theta^l, \Theta^r, \kappa, \Omega^l, \Omega^r)$ is an \mathbf{S} -2-stack in 2-groupoids \mathbb{P} endowed with a structure of left \mathbb{G} -torsor $(\mathbb{P}, m^l, \mu^l, \Theta^l)$, with a structure of right \mathbb{G} -torsor $(\mathbb{P}, m^r, \mu^r, \Theta^r)$, with a natural 2-transformation κ which expresses the compatibility between the left and the right action of \mathbb{G} on \mathbb{P} , and finally with two modification of \mathbf{S} -2-stacks Ω^l and Ω^r which express the obstruction to the compatibility between the natural 2-transformation κ and the natural 2-transformations μ^l and μ^r respectively. We require also that the two modification Ω^l and Ω^r satisfy some compatibility conditions. \mathbb{G} -torsors build a 3-category $\text{TORS}(\mathbb{G})$ where the objects are \mathbb{G} -torsors, the 1-arrows are morphisms of \mathbb{G} -torsors, the 2-arrows are 2-morphisms of \mathbb{G} -torsors and the 3-arrows are 3-morphisms of \mathbb{G} -torsors (Definition 2.7, 2.9 2.10).

From now on we assume \mathbb{G} to be a Picard \mathbf{S} -2-stack. We define the following groups:

- $\text{TORS}^1(\mathbb{G})$ is the group of equivalence classes of objects of $\text{TORS}(\mathbb{G})$,
- $\text{TORS}^0(\mathbb{G})$ is the group of 2-isomorphism classes of morphisms of \mathbb{G} -torsors from a \mathbb{G} -torsors \mathbb{P} to itself,
- $\text{TORS}^{-1}(\mathbb{G})$ is the group of 3-isomorphism classes of 2-automorphisms of morphisms of \mathbb{G} -torsors from \mathbb{P} to itself, and finally
- $\text{TORS}^{-2}(\mathbb{G})$ is the group of 3-automorphisms of the 2-automorphisms of morphisms of \mathbb{G} -torsors from \mathbb{P} to itself.

The group structures on the sets $\text{TORS}^i(\mathbb{G})$ are defined in the following way: Using push-downs of \mathbf{S} -2-stacks in 2-groupoids, we introduce the notion of *contracted product* of \mathbb{G} -torsors (Definition 2.12) which furnishes the abelian group structure on $\text{TORS}^1(\mathbb{G})$ (Proposition

2.13). The 2-groupoid $\mathrm{Hom}_{\mathrm{TORS}(\mathbb{G})}(\mathbb{P}, \mathbb{P})$ of morphisms of \mathbb{G} -torsors from a \mathbb{G} -torsor \mathbb{P} to itself is endowed with a Picard 2-stack structure (Lemma 3.1) and so its homotopy groups $\pi_i(\mathrm{Hom}_{\mathrm{TORS}(\mathbb{G})}(\mathbb{P}, \mathbb{P}))$ for $i = 0, 1, 2$ are abelian groups. Since by definition

$$\mathrm{TORS}^{-i}(\mathbb{G}) \cong \pi_i(\mathrm{Hom}_{\mathrm{TORS}(\mathbb{G})}(\mathbb{P}, \mathbb{P}))$$

we have that the sets $\mathrm{TORS}^i(\mathbb{G})$ for $i = 0, -1, -2$ are abelian groups.

If K is a complex of abelian sheaves over \mathbf{S} , we denote by $H^i(K)$ the cohomology group $H^i(\mathrm{R}\Gamma(K))$ of the derived functor of the functor of global sections applied to K . With these notation, we can finally state our first Theorem, which can be read from left to right and from right to left, furnishing respectively

- (1) a parametrization of the elements of $\mathrm{TORS}^i(\mathbb{G})$ by the cohomology group $H^i([\mathbb{G}]^{\mathrm{bb}})$, and so in particular a parametrization of the equivalence classes of \mathbb{G} -torsors by the cohomology group $H^1([\mathbb{G}]^{\mathrm{bb}})$,
- (2) a geometrical description of the cohomology groups $H^i(-)$ of length 3 complexes of abelian sheaves via torsors under Picard \mathbf{S} -2-stacks.

Theorem 0.1. *Let \mathbb{G} be a Picard \mathbf{S} -2-stack. Then we have the following isomorphisms*

$$\mathrm{TORS}^i(\mathbb{G}) \cong H^i([\mathbb{G}]^{\mathrm{bb}}) \quad \text{for } i = 1, 0, -1, -2.$$

Gr- \mathbf{S} -3-stacks are not defined yet. Assuming their existence, the contracted product of \mathbb{G} -torsors, which equippes the set $\mathrm{TORS}^1(\mathbb{G})$ of equivalence classes of \mathbb{G} -torsors with an abelian group law, should define a structure of gr- \mathbf{S} -3-stack on the 3-category $\mathrm{TORS}(\mathbb{G})$. In this setting our Theorem 0.1 says that the 3-category $\mathrm{TORS}(\mathbb{G})$ of \mathbb{G} -torsors should be actually the gr- \mathbf{S} -3-stack associated to the object of $\mathcal{D}^{[-3,0]}(\mathbf{S})$

$$\tau_{\leq 0} \mathrm{R}\Gamma([\mathbb{G}]^{\mathrm{bb}})$$

via the generalization of the equivalence $2\mathrm{st}^{\mathrm{bb}}$ (0.1) to gr- \mathbf{S} -3-stacks and to length 4 complexes of sheaves of sets on \mathbf{S} . More generally, we expect that if \mathbf{G} is a gr- \mathbf{S} - n -stacks, \mathbf{G} -torsors build a gr- \mathbf{S} -($n+1$)-stack which should be equivalent to the gr- \mathbf{S} -($n+1$)-stack associated to the object $\tau_{\leq 0} \mathrm{R}\Gamma([\mathbf{G}]^{\mathrm{bb}})$ via the generalization of the equivalence $2\mathrm{st}^{\mathrm{bb}}$ (0.1) to gr- \mathbf{S} -($n+1$)-stacks and to length $n+2$ complexes of sheaves of sets. Moreover, always in the setting of gr- \mathbf{S} -3-stacks, in order to define the groups $\mathrm{TORS}^i(\mathbb{G})$ we could use the homotopy groups π_i (for $i = 0, 1, 2, 3$) of the gr- \mathbf{S} -3-stack $\mathrm{TORS}(\mathbb{G})$: in fact we have

$$\mathrm{TORS}^i(\mathbb{G}) = \pi_{-i+1}(\mathrm{TORS}(\mathbb{G})) \quad \text{for } i = 1, 0, -1, -2.$$

If \mathbb{P} and \mathbb{G} are two Picard \mathbf{S} -2-stacks, an extension $(\mathbb{E}, I, J, \varepsilon)$ of \mathbb{P} by \mathbb{G} consists of a Picard \mathbf{S} -2-stack \mathbb{E} , two additive 2-functors $I : \mathbb{G} \rightarrow \mathbb{E}$ and $J : \mathbb{E} \rightarrow \mathbb{P}$, and a morphism of additive 2-functors $\varepsilon : J \circ I \Rightarrow 0$, such that the following equivalent conditions are satisfied:

- $\pi_0(J) : \pi_0(\mathbb{E}) \rightarrow \pi_0(\mathbb{P})$ is surjective and I induces an equivalence of Picard \mathbf{S} -2-stacks between \mathbb{G} and $\mathrm{Ker}(J)$,
- $\pi_2(I) : \pi_2(\mathbb{G}) \rightarrow \pi_2(\mathbb{E})$ is injective and J induces an equivalence of Picard \mathbf{S} -2-stacks between $\mathrm{Coker}(I)$ and \mathbb{P} .

In [3] we have proved that extensions of \mathbb{P} by \mathbb{G} form a 3-category $\mathcal{E}\mathrm{xt}(\mathbb{P}, \mathbb{G})$ and we have computed the homotopy groups $\pi_i(\mathcal{E}\mathrm{xt}(\mathbb{P}, \mathbb{G}))$ for $i = 0, 1, 2, 3$. In this paper, we describe extensions of Picard \mathbf{S} -2-stacks in terms of torsors under Picard \mathbf{S} -2-stacks. We start with a special case of extensions, which involve a Picard \mathbf{S} -2-stack generated by an \mathbf{S} -2-stack in 2-groupoids (see Definition 3.4), and whose description in terms of torsors is a direct consequence of Theorem 0.1:

Corollary 0.2. *Let \mathbb{G} be a Picard \mathbf{S} -2-stacks. Consider a $gr\text{-}\mathbf{S}$ -2-stack \mathbb{Q} , associated to a length 3 complex of sheaves of groups, and the Picard \mathbf{S} -2-stack $\mathbb{Z}[\mathbb{Q}]$ generated it. We have the following tri-equivalence of 3-categories*

$$\mathcal{E}xt(\mathbb{Z}[\mathbb{Q}], \mathbb{G}) \cong \text{TORS}(\mathbb{G}_{\mathbb{Q}})$$

where $\text{TORS}(\mathbb{G}_{\mathbb{Q}})$ denotes the 3-category of $\mathbb{G}_{\mathbb{Q}}$ -torsors over \mathbb{Q} (see Definition 2.16).

Now, for the general case, if \mathbb{P} and \mathbb{G} are two Picard \mathbf{S} -2-stacks, we find an explicit description of extensions of \mathbb{P} by \mathbb{G} in terms of $\mathbb{G}_{\mathbb{P}}$ -torsors over \mathbb{P} which are endowed with an abelian group law on the fibers. More precisely, it exists a tri-equivalence of 3-categories between the 3-category $\mathcal{E}xt(\mathbb{P}, \mathbb{G})$ and the 3-category consisting of the data $(\mathbb{E}, M, \alpha, \mathbf{a}, \chi, \mathbf{s}, \mathbf{c}_1, \mathbf{c}_2, \mathbf{p})$, where \mathbb{E} is a $\mathbb{G}_{\mathbb{P}}$ -torsors over \mathbb{P} , $M : p_1^* \mathbb{E} \wedge p_2^* \mathbb{E} \rightarrow \otimes^* \mathbb{E}$ is a morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors over $\mathbb{P} \times \mathbb{P}$ defining a group law on the fibers of \mathbb{E} (here \otimes is the group law of \mathbb{P} and $p_i : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ are the projections), α is a 2-morphism of $\mathbb{G}_{\mathbb{P}^3}$ -torsors expressing the associativity constraint of this group law defined by M , χ is a 2-morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors expressing the braiding constraint of this group law defined by M , and finally $\mathbf{a}, \mathbf{s}, \mathbf{c}_1, \mathbf{c}_2, \mathbf{p}$ are 3-morphisms of $\mathbb{G}_{\mathbb{P}^i}$ -torsors (with $i = 4, 2, 3, 3, 1$ respectively) expressing respectively the obstruction to the coherence of α , the obstruction to the coherence of χ , the obstruction to the compatibility between α and χ and the obstruction to the strictness of χ . We require also that these 3-morphisms of $\mathbb{G}_{\mathbb{P}^i}$ -torsors satisfy some coherence and compatibility conditions. Summarizing we have

Theorem 0.3. *Let \mathbb{P} and \mathbb{G} be two Picard \mathbf{S} -2-stacks. Then we have the following tri-equivalence of 3-categories*

$$\mathcal{E}xt(\mathbb{P}, \mathbb{G}) \simeq \left\{ \begin{array}{l} (\mathbb{E}, M, \alpha, \mathbf{a}, \chi, \mathbf{s}, \mathbf{c}_1, \mathbf{c}_2, \mathbf{p}) \mid \mathbb{E} = \mathbb{G}_{\mathbb{P}} - \text{torsor over } \mathbb{P}, \\ M : p_1^* \mathbb{E} \wedge p_2^* \mathbb{E} \rightarrow \otimes^* \mathbb{E}, \alpha, \mathbf{a}, \chi, \mathbf{s}, \mathbf{c}_1, \mathbf{c}_2, \mathbf{p} \text{ described in Prop.4.1} \end{array} \right\}$$

This Theorem generalizes to Picard \mathbf{S} -2-stacks the following result of Grothendieck in [9, Exposé VII 1.1.6 and 1.2]: if P and G are two abelian sheaves, to have an extension of P by G is the same thing as to have a G_P -torsor E over P , and an isomorphism $pr_1^* E \rightarrow +^* E$ of G_{P^2} -torsors over $P \times P$ satisfying some associativity and commutativity conditions.

As a consequence of the description of extensions of Picard \mathbf{S} -2-stacks in term of torsors, we get a right resolution of the 3-category $\mathcal{E}xt(\mathbb{P}, \mathbb{G})$ by 3-categories of $\mathbb{G}_{\mathbb{P}^i}$ -torsors for $i \in \{1, 2, 3, 4, 5\}$: in fact, as a direct consequence of Corollary 0.2 and of Theorem 0.3, we have

Corollary 0.4. *Let \mathbb{P} and \mathbb{G} be two Picard \mathbf{S} -2-stacks. The complex*

$$\begin{aligned} 0 \rightarrow \text{TORS}(\mathbb{G}_{\mathbb{P}}) \xrightarrow{D_0^*} \text{TORS}(\mathbb{G}_{\mathbb{P}^2}) \xrightarrow{D_1^*} \text{TORS}(\mathbb{G}_{\mathbb{P}^2}) \times \text{TORS}(\mathbb{G}_{\mathbb{P}^3}) \xrightarrow{D_2^*} \text{TORS}(\mathbb{G}_{\mathbb{P}^4}) \times \text{TORS}(\mathbb{G}_{\mathbb{P}^3}) \xrightarrow{D_3^*} \dots \\ \dots \xrightarrow{D_3^*} \text{TORS}(\mathbb{G}_{\mathbb{P}^5}) \times \text{TORS}(\mathbb{G}_{\mathbb{P}^4}) \rightarrow 0 \end{aligned}$$

is a right resolution of the 3-category $\mathcal{E}xt(\mathbb{P}, \mathbb{G})$ of extensions of \mathbb{P} by \mathbb{G} . The differential operators D_i are defined in (5.3) and D_i^* denotes the pull-back via the differential operator D_i (for $i = 0, 1, 2, 3$).

This last result can be rewritten in the derived category $\mathcal{D}(\mathbf{S})$ of abelian sheaves on \mathbf{S} : using the homological interpretation of extensions of Picard \mathbf{S} -2-stacks proved in [3, Theorem 1.1] and using the homological interpretation of torsors under Picard \mathbf{S} -2-stacks proved in Theorem 0.1, we obtain in $\mathcal{D}(\mathbf{S})$ a right resolution of the object $\text{RHom}(-, -)$ applied to length 3 complexes by objects $\text{R}\Gamma(-)$ applied to length 3 complexes:

Corollary 0.5. *Let P and G be length 3 complexes of abelian sheaves on \mathbf{S} . The complex*

$$0 \rightarrow \tau_{\leq 0} \mathrm{R}\Gamma(G_P) \xrightarrow{d_0} \tau_{\leq 0} \mathrm{R}\Gamma(G_{P^2}) \xrightarrow{d_1} \tau_{\leq 0} \mathrm{R}\Gamma(G_{P^2}) \times \tau_{\leq 0} \mathrm{R}\Gamma(G_{P^3}) \xrightarrow{d_2} \mathrm{R}\Gamma(G_{P^4}) \times \tau_{\leq 0} \mathrm{R}\Gamma(G_{P^3}) \xrightarrow{d_3} \dots \\ \dots \xrightarrow{d_3} \tau_{\leq 0} \mathrm{R}\Gamma(G_{P^5}) \times \tau_{\leq 0} \mathrm{R}\Gamma(G_{P^4}) \rightarrow 0$$

is a right resolution of the object $\tau_{\leq 0} \mathrm{R}\mathrm{Hom}(P, G[1])$ of $\mathcal{D}^{[-3,0]}(\mathbf{S})$.

The study of torsors under Picard \mathbf{S} -2-stacks is a first step toward the theory of biextensions of Picard \mathbf{S} -2-stacks: in fact, if \mathbb{P}, \mathbb{Q} and \mathbb{G} are Picard \mathbf{S} -2-stacks, a biextension of (\mathbb{P}, \mathbb{Q}) by \mathbb{G} is a $\mathbb{G}_{\mathbb{P} \times \mathbb{Q}}$ -torsor over $\mathbb{P} \times \mathbb{Q}$ endowed with two compatible group laws on the fibers. Using the canonical flat partial resolution $\mathbb{L}(\mathbb{P})$ of \mathbb{P} (5.2) and the 3-category $\Psi_{\mathbb{L}(\mathbb{P}) \otimes \mathbb{L}(\mathbb{Q})}(\mathbb{G})$ introduced in Definition 5.1, we get easily the homological interpretation of biextensions of (\mathbb{P}, \mathbb{Q}) by \mathbb{G} : $\pi_{-i+1}(\mathrm{Biext}(\mathbb{P}, \mathbb{Q}; \mathbb{G})) \cong \mathrm{Hom}_{\mathcal{D}(\mathbf{S})}([\mathbb{P}]^{bb} \otimes [\mathbb{Q}]^{bb}, [\mathbb{G}]^{bb}[i])$ for $i = 1, 0, -1, -2$, where $\pi_{-i+1}(\mathrm{Biext}(\mathbb{P}, \mathbb{Q}; \mathbb{G}))$ are the homotopy groups of the 3-category of biextensions of (\mathbb{P}, \mathbb{Q}) by \mathbb{G} . The theory of biextensions has important applications in the theory of motives since biextensions define bilinear morphisms between motives.

We hope that this work will shed some light on the notions of “torsor” for higher categories with group-like operation. In particular, as in [3], we pay a lot of attention to write down the proofs in such a way that they can be easily generalized to Picard \mathbf{S} - n -stacks and to length $n+1$ complexes of abelian sheaves on \mathbf{S} .

The most relevant ancestor of this paper is [2] where the first author describes explicitly extensions of Picard \mathbf{S} -stacks in terms of torsors under Picard \mathbf{S} -stacks which are endowed with an abelian group law on the fibers (see in particular [2, Theorem 4.1]). In order to generalize from \mathbf{S} -stacks to \mathbf{S} -2-stacks the notions of [2] that we need in this paper (as, for example, the definition of torsor) we proceed as follows: the data involving 1-arrows and 2-arrows remain the same, but the coherence axioms or the compatibility conditions, that 2-arrows have to satisfy and that are given via equations of 1-arrows, are replaced by 3-arrows which express the obstruction to the above coherence axioms or compatibility conditions for 2-arrows and we require that these 3-arrows satisfies some coherence axioms or compatibility conditions that are given via equations of 2-arrows. This way to generalize mathematical notions to higher categories can be observed clearly looking to the differential operators of the resolution of $\mathcal{E}\mathrm{xt}(\mathbb{P}, \mathbb{G})$ in Corollary 0.4: the differential operator D_0 involves 1-arrows, D_1 involves 2-arrows, D_2 involves 3-arrows, and finally D_3 involves the coherence axioms or compatibility conditions that 3-arrows have to satisfy and that are given via equations of 2-arrows. If $\mathbb{P} = P$ and $\mathbb{G} = G$ are just abelian sheaves on \mathbf{S} (this is the case treated by Grothendieck in [9, Exposé VII 3.5]), we have just the differential operators D_0 and D_1 and the resolution of $\mathcal{E}\mathrm{xt}(P, G)$ finishes with the term $\mathrm{TORS}(G_{P^2}) \times \mathrm{TORS}(G_{P^3})$. If $\mathbb{P} = \mathcal{P}$ and $\mathbb{G} = \mathcal{G}$ are Picard \mathbf{S} -stacks (this is the case done by the first author in [2§9]), we have the differential operators D_0, D_1 and D_2 and the resolution of $\mathcal{E}\mathrm{xt}(\mathcal{P}, \mathcal{G})$ finishes with $\mathrm{TORS}(\mathcal{G}_{\mathcal{P}^4}) \times \mathrm{TORS}(\mathcal{G}_{\mathcal{P}^3})$. If we go one higher dimension, i.e. in the case where \mathbb{P} and \mathbb{G} are Picard \mathbf{S} -2-stacks, we have to add the differential operator D_3 which involves the coherence axioms or compatibility conditions that 3-arrows have to satisfy and that are given via equations of 2-arrows.

ACKNOWLEDGMENT

The second author is supported by KFUPM under research grant RG1322-1 and RG 1322-2.

NOTATION

For the notions of strict 2-category (just called 2-category), 2-groupoid, bicategory, strict 3-category (just called 3-category), tricategory and triequivalence of tricategories, we refer to the section NOTATION of our previous paper [3].

Let \mathbf{S} be a site. For the notions of \mathbf{S} -pre-stacks, \mathbf{S} -stacks and morphisms of \mathbf{S} -stacks we refer to [7, Chapter II 1.2]. An \mathbf{S} -2-stack in 2-groupoids \mathbb{P} is a fibered 2-category in 2-groupoids over \mathbf{S} such that for every pair of objects X, Y of the 2-category $\mathbb{P}(U)$ with U an object of \mathbf{S} , the fibered category of morphisms $\text{Arr}_{\mathbb{P}(U)}(X, Y)$ of $\mathbb{P}(U)$ is an \mathbf{S}/U -stack (called the \mathbf{S}/U -stack of morphisms), and 2-descent is effective for objects in \mathbb{P} . For the notions of morphisms of \mathbf{S} -2-stacks, natural 2-transformations of \mathbf{S} -2-stacks and modifications of \mathbf{S} -2-stacks we refer to [6§6].

If \mathbb{P} and \mathbb{Q} are two \mathbf{S} -2-stacks in 2-groupoids, the product of \mathbb{P} and \mathbb{Q} , denoted by $\mathbb{P} \times \mathbb{Q}$, is the \mathbf{S} -2-stack in 2-groupoids defined as follows: for any object U of \mathbf{S} , an object of $(\mathbb{P} \times \mathbb{Q})(U)$ is a pair (p, q) with p an object of $\mathbb{P}(U)$ and q an object of $\mathbb{Q}(U)$. In this paper we denote by \mathbb{P}^n the product of n copies of \mathbb{P} , with $n \in \mathbb{N}$. Moreover, $\mathbf{0}$ will be the Picard \mathbf{S} -2-stack whose only object is the neutral object and whose only 1- and 2-arrows are the identities.

Denote by $\mathcal{K}(\mathbf{S})$ the category of complexes of abelian sheaves on the site \mathbf{S} : all complexes that we consider in this paper are cochain complexes. Let $\mathcal{K}^{[-2,0]}(\mathbf{S})$ be the subcategory of $\mathcal{K}(\mathbf{S})$ consisting of complexes $K = (K^i)_{i \in \mathbb{Z}}$ such that $K^i = 0$ for $i \neq -2, -1$ or 0 . The good truncation $\tau_{\leq n} K$ of a complex K of $\mathcal{K}(\mathbf{S})$ is the following complex: $(\tau_{\leq n} K)^i = K^i$ for $i < n$, $(\tau_{\leq n} K)^n = \ker(d^n)$ and $(\tau_{\leq n} K)^i = 0$ for $i > n$. The bad truncation $\sigma_{\leq n} K$ of a complex K of $\mathcal{K}(\mathbf{S})$ is the following complex: $(\sigma_{\leq n} K)^i = K^i$ for $i \leq n$ and $(\sigma_{\leq n} K)^i = 0$ for $i > n$. For any $i \in \mathbb{Z}$, the shift functor $[i] : \mathcal{K}(\mathbf{S}) \rightarrow \mathcal{K}(\mathbf{S})$ acts on a complex $K = (K^n)_{n \in \mathbb{Z}}$ as $(K[i])^n = K^{i+n}$ and $d_{K[i]}^n = (-1)^i d_K^{n+i}$.

Denote by $\mathcal{D}(\mathbf{S})$ the derived category of the category of abelian sheaves on \mathbf{S} , and let $\mathcal{D}^{[-2,0]}(\mathbf{S})$ be the full subcategory of $\mathcal{D}(\mathbf{S})$ consisting of complexes K such that $H^i(K) = 0$ for $i \neq -2, -1$ or 0 . If K and L are complexes of $\mathcal{D}(\mathbf{S})$, the group $\text{Ext}^i(K, L)$ is by definition $\text{Hom}_{\mathcal{D}(\mathbf{S})}(K, L[i])$ for any $i \in \mathbb{Z}$. Let $\text{RHom}(-, -)$ be the derived functor of the bifunctor $\text{Hom}(-, -)$. The cohomology groups $H^i(\text{RHom}(K, L))$ of $\text{RHom}(K, L)$ are isomorphic to $\text{Hom}_{\mathcal{D}(\mathbf{S})}(K, L[i])$. The functor $\Gamma(-)$ of global sections is isomorphic to the functor $\text{Hom}(\mathbf{e}, -)$, where \mathbf{e} is the final object of the category of abelian sheaves on \mathbf{S} . Let $\text{R}\Gamma(-)$ be the derived functor of the functor $\Gamma(-)$ of global sections. We denote the cohomology groups $H^i(\text{R}\Gamma(K))$ of $\text{R}\Gamma(K)$ by $H^i(K)$.

1. RECALL ON THE 3-CATEGORY OF PICARD 2-STACKS

In this paper we need the explicit description of the data underlying the notion of Picard 2-stack and also of the constraints that those data have to satisfy. Therefore we start recalling them. Let \mathbf{S} be a site. In the following definitions U will denote an object of \mathbf{S} . Moreover in the diagrams involving 2-arrows, we will put the symbol \cong in the cells which commute up to a modification of \mathbf{S} -2-stacks coming from the Picard structure.

A strict Picard \mathbf{S} -2-stack (just called Picard \mathbf{S} -2-stack) $\mathbb{P} = (\mathbb{P}, \otimes, \mathbf{a}, \pi, \mathbf{c}, \zeta, \mathfrak{h}_1, \mathfrak{h}_2, \eta)$ is an \mathbf{S} -2-stack in 2-groupoids \mathbb{P} equipped with

- (1) a morphism of \mathbf{S} -2-stacks $\otimes : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$, called the *group law* of \mathbb{P} . For simplicity instead of $X \otimes Y$ we write just XY for all $X, Y \in \mathbb{P}(U)$;
- (2) two natural 2-transformations of \mathbf{S} -2-stacks $\mathbf{a} : \otimes \circ (\otimes \times \text{id}_{\mathbb{P}}) \Rightarrow \otimes \circ (\text{id}_{\mathbb{P}} \times \otimes)$, called the *associativity*, and $\mathbf{c} : \otimes \circ \mathbf{s} \Rightarrow \otimes$ with $\mathbf{s}(X, Y) = (Y, X)$ for all $X, Y \in \mathbb{P}(U)$, called the *braiding*, which express respectively the associativity and the commutativity constraints of the group law \otimes of \mathbb{P} ;

- (3) a modification π of **S**-2-stacks whose component at $(X, Y, Z, W) \in \mathbb{P}^4(U)$ is the 2-arrow

$$(1.1) \quad \begin{array}{ccc} & ((XY)Z)W & \\ \swarrow \scriptstyle a_{(XY,Z,W)} & & \searrow \scriptstyle a_{(X,Y,Z)}W \\ (XY)(ZW) & \xleftarrow{\scriptstyle \pi_{(X,Y,Z,W)}} & (X(YZ))W \\ \downarrow \scriptstyle a_{(X,Y,ZW)} & & \downarrow \scriptstyle a_{(X,YZ,W)} \\ X(Y(ZW)) & \xleftarrow{\scriptstyle Xa_{(Y,Z,W)}} & X((YZ)W) \end{array}$$

and which expresses the obstruction to the coherence of the associativity a (i.e. the obstruction to the pentagonal axiom);

- (4) a modification ζ of **S**-2-stacks whose component at $(X, Y) \in \mathbb{P}^2(U)$ is the 2-arrow

$$(1.2) \quad \zeta_{(X,Y)} : \text{id}_{XY} \Rightarrow c_{(Y,X)} \circ c_{(X,Y)}$$

and which expresses the obstruction to the coherence of the braiding c . The modification ζ implies the weak invertability of the braiding c ;

- (5) two modifications $\mathfrak{h}_1, \mathfrak{h}_2$ of **S**-2-stacks whose components at $(X, Y, Z) \in \mathbb{P}^3(U)$ are the 2-arrows

$$(1.3) \quad \begin{array}{ccc} & X(YZ) \xrightarrow{c_{(X,Y,Z)}} (YZ)X & \\ \swarrow \scriptstyle a_{(X,Y,Z)} & & \searrow \scriptstyle a_{(Y,Z,X)} \\ (XY)Z & \Downarrow \scriptstyle \mathfrak{h}_1(X,Y,Z) & Y(ZX) \\ \searrow \scriptstyle c_{(X,Y)}Z & & \swarrow \scriptstyle Yc_{(X,Z)} \\ (YX)Z & \xrightarrow{\scriptstyle a_{(Y,X,Z)}} & Y(XZ) \end{array} \quad \begin{array}{ccc} & (XY)Z \xrightarrow{c_{(Z,XY)}^{-1}} Z(XY) & \\ \swarrow \scriptstyle a_{(X,Y,Z)}^{-1} & & \searrow \scriptstyle a_{(Z,X,Y)}^{-1} \\ X(YZ) & \Downarrow \scriptstyle \mathfrak{h}_2(X,Y,Z) & (ZX)Y \\ \searrow \scriptstyle Xc_{(Z,Y)}^{-1} & & \swarrow \scriptstyle c_{(Z,X)}^{-1}Y \\ X(ZY) & \xrightarrow{\scriptstyle a_{(X,Z,Y)}^{-1}} & (XZ)Y \end{array}$$

and which express the obstruction to the compatibility between the associativity a and the braiding c (i.e. the obstruction to the hexagonal axiom). The diagram defining $\mathfrak{h}_2(X,Y,Z)$ is just obtained from the diagram defining $\mathfrak{h}_1(X,Y,Z)$ by inverting all the arrows and by replacing X with Z , Y with X and Z with Y ;

- (6) a modification η of **S**-2-stacks whose component at $X \in \mathbb{P}(U)$ is the 2-arrow

$$(1.4) \quad \eta_X : c_{(X,X)} \Rightarrow \text{id}_{XX}$$

and which expresses the obstruction to the strictness of the braiding c .

These data satisfy the following compatibility conditions:

- (i) for any $X \in \mathbb{P}(U)$, the morphism of **S**-2-stacks $X \otimes - : \mathbb{P} \rightarrow \mathbb{P}$, called the left multiplication by X , is an equivalence of **S**-2-stacks;

- (ii) the modification π is coherent, i.e. it satisfies the coherence axiom of Stasheff's polytope (see [11§4]): for all $X, Y, Z, W, T \in \mathbb{P}(U)$ the following equation of 2-arrows holds

$$(1.5) \quad \begin{array}{c} \begin{array}{ccccc} & & X(Y(Z(WT))) & & \\ & \nearrow & & \nwarrow & \\ (XY)(Z(WT)) & & & & X(Y((ZW)T)) \\ \uparrow & \Leftarrow & \uparrow & \Leftarrow & \uparrow \\ ((XY)Z)(WT) & X((YZ)(WT)) & X((Y(ZW))T) & & \\ \uparrow & \searrow & \uparrow & \swarrow & \uparrow \\ (((XY)Z)W)T \cong (X(YZ))(WT) & X(((YZ)W)T) & & & \\ \downarrow & \nearrow & \uparrow & \searrow & \uparrow \\ ((X(YZ))W)T & \xrightarrow{\pi(X,Y,Z,W,T)} & (X((YZ)W))T & & \end{array} \\ \\ \end{array} = \begin{array}{c} \begin{array}{ccccc} & & X(Y(Z(WT))) & & \\ & \nearrow & & \nwarrow & \\ (XY)(Z(WT)) & \cong & & & X(Y((ZW)T)) \\ \uparrow & \nwarrow & \uparrow & \nwarrow & \uparrow \\ ((XY)Z)(WT) & (XY)((ZW)T) & & & X((Y(ZW))T) \\ \uparrow & \nwarrow & \uparrow & \nwarrow & \uparrow \\ (((XY)Z)W)T & ((XY)(ZW))T & X((Y(ZW))T) & & \\ \downarrow & \nearrow & \uparrow & \searrow & \uparrow \\ ((X(YZ))W)T & \xrightarrow{\pi(X,Y,Z,W,T)} & (X((YZ)W))T & & \end{array} \end{array}$$

- (iii) the modification ζ is coherent, i.e. for all $X, Y, Z \in \mathbb{P}(U)$ the below equation of 2-arrows holds

$$(1.6) \quad XY \xrightarrow{c_{(X,Y)}} YX \begin{array}{c} \xrightarrow{\text{id}_{YX}} \\ \Downarrow \zeta_{(Y,X)} \\ \xrightarrow{c_{(X,Y)} \circ c_{(Y,X)}} \end{array} YX = XY \begin{array}{c} \xrightarrow{\text{id}_{XY}} \\ \Downarrow \zeta_{(X,Y)} \\ \xrightarrow{c_{(Y,X)} \circ c_{(X,Y)}} \end{array} XY \xrightarrow{c_{(X,Y)}} YX$$

Together with its coherence constraint, ζ is the natural generalization to Picard **S**-2-stacks of the condition $c \circ c = \text{id}_{\mathbb{P}}$ which Picard **S**-stacks satisfy;

- (iv) the modification \mathfrak{h}_1 is compatible with π , i.e. for all $X, Y, Z, W \in \mathbb{P}(U)$ the following equation of 2-arrows is satisfied

$$(1.7) \quad \begin{array}{c} \begin{array}{ccccc} & & (X(YZ))W \longrightarrow X((YZ)W) & & \\ & \nearrow & \Downarrow \pi_{(X,Y,Z,W)} & \nwarrow & \\ ((XY)Z)W & \longrightarrow & (XY)(ZW) & \longrightarrow & X(Y(ZW)) \\ \downarrow & \cong & \downarrow & \downarrow & \downarrow \\ ((YX)Z)W & \longrightarrow & (YX)(ZW) & \xrightarrow{\mathfrak{h}_1(X,Y,Z,W)} & (Y(ZW))X \\ \downarrow & \nearrow & \downarrow & \searrow & \downarrow \\ (Y(XZ))W & \xrightarrow{\pi(Y,X,Z,W)} & Y(X(ZW)) & \longrightarrow & Y((ZW)X) \\ \downarrow & \nearrow & \downarrow & \searrow & \downarrow \\ Y((XZ)W) & \xrightarrow{\mathfrak{h}_1(X,Z,W)} & Y(Z(XW)) & & \end{array} \\ \\ \end{array} = \begin{array}{c} \begin{array}{ccccc} & & (X(YZ))W \longrightarrow X((YZ)W) & & \\ & \nearrow & \downarrow & \nwarrow & \\ ((XY)Z)W & \longrightarrow & ((YX)Z)W & \longrightarrow & ((YX)Z)W \\ \downarrow & \searrow & \downarrow & \searrow & \downarrow \\ ((YX)Z)W & \xrightarrow{\mathfrak{h}_1(X,Y,Z,W)} & (Y(XZ))W & \xrightarrow{\mathfrak{h}_1(X,Y,Z,W)} & (Y(XZ))W \\ \downarrow & \nearrow & \downarrow & \searrow & \downarrow \\ (Y(XZ))W & \xrightarrow{\pi(Y,X,Z,W)} & Y(X(ZW)) & \longrightarrow & Y((ZW)X) \\ \downarrow & \nearrow & \downarrow & \searrow & \downarrow \\ Y((XZ)W) & \xrightarrow{\mathfrak{h}_1(X,Z,W)} & Y(Z(XW)) & & \end{array} \end{array}$$

Moreover the modifications \mathfrak{h}_1 and \mathfrak{h}_2 are comparable, i.e. the pasting of the 2-arrows in the diagram below, denoted by $\mathfrak{h}_1 \square \mathfrak{h}_2$, is the identity:

(1.8)

and similarly, the pasting of the 2-arrows in the diagram below, denoted by $\mathfrak{h}_2 \square \mathfrak{h}_1$, is the identity:

(1.9)

We require also that \mathfrak{h}_1 and \mathfrak{h}_2 are compatible under this comparison, i.e. $(\mathfrak{h}_1 \square \mathfrak{h}_2) \square \mathfrak{h}_1 = \mathfrak{h}_1 \square (\mathfrak{h}_2 \square \mathfrak{h}_1) = \mathfrak{h}_1$ and $(\mathfrak{h}_2 \square \mathfrak{h}_1) \square \mathfrak{h}_2 = \mathfrak{h}_2 \square (\mathfrak{h}_1 \square \mathfrak{h}_2) = \mathfrak{h}_2$.

Finally using the terminology of Kapranov and Voevodsky in [11], we require that the 2-arrows defining the Z-systems coincide, i.e. for all $X, Y, Z \in \mathbb{P}(U)$ the following equation

of 2-arrows holds

$$(1.10) \quad \begin{array}{c} \begin{array}{ccccc} & & (XZ)Y & & \\ & \nearrow & & \searrow & \\ & X(ZY) & & (ZX)Y & \\ & \nearrow & & \searrow & \\ (XY)Z & & X(YZ) & \xrightarrow{\mathfrak{h}_1^{-1}} & Z(XY) \\ & \searrow & & \nearrow & \\ & (YX)Z & & (ZY)X & \\ & \searrow & & \nearrow & \\ & Y(XZ) & & (YZ)X & \\ & & Y(ZX) & & \end{array} \\ \cong \\ \begin{array}{ccccc} & & (XZ)Y & & \\ & \nearrow & & \searrow & \\ & X(ZY) & & (ZX)Y & \\ & \nearrow & & \searrow & \\ (XY)Z & & X(YZ) & \xrightarrow{\mathfrak{h}_2^{-1}} & Z(XY) \\ & \searrow & & \nearrow & \\ & (YX)Z & & (ZY)X & \\ & \searrow & & \nearrow & \\ & Y(XZ) & & (YZ)X & \\ & & Y(ZX) & & \end{array} \end{array} = \begin{array}{c} \begin{array}{ccccc} & & (XZ)Y & & \\ & \nearrow & & \searrow & \\ & X(ZY) & & (ZX)Y & \\ & \nearrow & & \searrow & \\ (XY)Z & & X(YZ) & \xrightarrow{\mathfrak{h}_2} & Z(XY) \\ & \searrow & & \nearrow & \\ & (YX)Z & & (ZY)X & \\ & \searrow & & \nearrow & \\ & Y(XZ) & & (YZ)X & \\ & & Y(ZX) & & \end{array} \end{array}$$

The last three conditions (1.8), (1.9), (1.10) satisfied by the modifications \mathfrak{h}_1 and \mathfrak{h}_2 allow us to see \mathfrak{h}_2 as a kind of inverse for \mathfrak{h}_1 .

- (v) the modification η satisfies the following two compatibility conditions: the first one is that $\eta * \eta = \zeta$, the second one is that for all $X, Y \in \mathbb{P}(U)$ there is an additive relation between η_X, η_Y and η_{XY} , i.e. η_{XY} is equal to the pasting of the 2-arrows in the following diagram

$$(1.11) \quad \begin{array}{c} \begin{array}{c} \uparrow \pi^* \\ \begin{array}{c} \xrightarrow{\quad} (X(YX))Y \\ \cong \searrow \nearrow \zeta \\ X((XY)Y) \xrightarrow{\quad} (X(XY))Y \\ \uparrow \pi^* \quad \uparrow \pi^* \\ \begin{array}{ccccc} X((YX)Y) & & X(X(YY)) & & ((XX)Y)Y \\ \nearrow & \searrow & \nearrow & \searrow & \nearrow \\ (XY)(XY) & \xleftarrow{\mathfrak{h}_1} & X(Y(XY)) & \xrightarrow{\left(\begin{smallmatrix} \Rightarrow \\ \eta \end{smallmatrix}\right)} & X(X(YY)) & \xrightarrow{\left(\begin{smallmatrix} \Leftarrow \\ \eta \end{smallmatrix}\right)} & ((XX)Y)Y & \xleftarrow{\mathfrak{h}_1} & ((XY)X)Y & \xleftarrow{\quad} (XY)(XY) \end{array} \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ X((XY)Y) & \xrightarrow{\quad} & (X(XY))Y \\ \uparrow \mathfrak{h}_2 \end{array} \end{array} \end{array}$$

Together with its coherence constraint, η is the natural generalization to Picard **S**-2-stacks of the condition $c_{(X,X)} = \text{id}_{XX}$ which Picard **S**-stacks satisfy. The modification η (which justifies the terminology of *strict Picard S-2-stack* of [5, Definition 8.5]) allows an unambiguous use of the operation $n \cdot a$: abelian groups and \mathbb{Z} -modules are the same thing, but at this higher level the second corresponds to “strict Picard **S**-2-stack”.

Remark 1.1. In [1], Aldrovandi and Tatar show that the condition (i) “for any $X \in \mathbb{P}(U)$, the left multiplication by X , $X \otimes - : \mathbb{P} \rightarrow \mathbb{P}$, is an equivalence of **S**-2-stacks” implies the existence of a pair (e_X, φ_X) consisting of an object $e_X \in \mathbb{P}(U)$ and a 1-arrow $\varphi_X : e_X \otimes e_X \rightarrow e_X$. In the same paper they also show that all pairs (e, φ) of this form are equivalent up to a unique 2-isomorphism. In [10], Joyal and Kock show that the 2-category of the pairs (e, φ) in a 2-monoidal category with strict associativity is equivalent to the 2-category

of classical units with left and right constraints compatible with each other. Their result generalizes without difficulty to Picard \mathbf{S} -2-stacks. Therefore any Picard \mathbf{S} -2-stack admits a global neutral object e endowed with two natural 2-transformations of \mathbf{S} -2-stacks

$$(1.12) \quad \mathbf{l} : e \otimes - \Rightarrow \text{id} \quad \text{and} \quad \mathbf{r} : - \otimes e \Rightarrow \text{id},$$

which express respectively the left and the right unit constraints and which satisfy some higher compatibility conditions (see [10]). Moreover the equivalence $X \otimes - : \mathbb{P} \rightarrow \mathbb{P}$ of \mathbf{S} -2-stacks implies also that any object of $\mathbb{P}(U)$ admits an inverse.

Let \mathbb{P} and \mathbb{Q} be two Picard \mathbf{S} -2-stacks. An additive 2-functor $(F, \gamma_F) : \mathbb{P} \rightarrow \mathbb{Q}$ is given by a morphism of \mathbf{S} -2-stacks $F : \mathbb{P} \rightarrow \mathbb{Q}$ (i.e. a cartesian 2-functor) and a natural 2-transformation of \mathbf{S} -2-stacks $\gamma_F : \otimes_{\mathbb{Q}} \circ F^2 \Rightarrow F \circ \otimes_{\mathbb{P}}$, which are compatible with the natural 2-transformations $\mathbf{a}_{\mathbb{P}}, \mathbf{c}_{\mathbb{P}}, \mathbf{a}_{\mathbb{Q}}, \mathbf{c}_{\mathbb{Q}}$.

Let $(F, \gamma_F), (G, \gamma_G) : \mathbb{P} \rightarrow \mathbb{Q}$ be additive 2-functors between Picard \mathbf{S} -2-stacks. A morphism of additive 2-functors $(\theta, \Gamma) : (F, \gamma_F) \Rightarrow (G, \gamma_G)$ is given by a natural 2-transformation of \mathbf{S} -2-stacks $\theta : F \Rightarrow G$ and a modification of \mathbf{S} -2-stacks $\Gamma : \gamma_G \circ \otimes_{\mathbb{Q}} * \theta^2 \Rightarrow \theta * \otimes_{\mathbb{P}} \circ \gamma_F$, so that θ and Γ are compatible with the additive structures of (F, γ_F) and (G, γ_G) .

Let $(\theta_1, \Gamma_1), (\theta_2, \Gamma_2) : (F, \gamma_F) \Rightarrow (G, \gamma_G)$ be morphisms of additive 2-functors. A modification of morphisms of additive 2-functors $\Sigma : (\theta_1, \Gamma_1) \Rightarrow (\theta_2, \Gamma_2)$ is given by a modification of \mathbf{S} -2-stacks $\Sigma : \theta_1 \Rightarrow \theta_2$ such that $(\Sigma * \otimes_{\mathbb{P}}) \gamma_F \circ \Gamma_1 = \Gamma_2 \circ \gamma_G (\otimes_{\mathbb{Q}} * \Sigma^2)$.

Picard \mathbf{S} -2-stacks over \mathbf{S} form a 3-category $2\text{PICARD}(\mathbf{S})$ whose objects are Picard \mathbf{S} -2-stacks and whose hom-2-groupoid consists of additive 2-functors, morphisms of additive 2-functors, and modifications of morphisms of additive 2-functors.

An equivalence of Picard \mathbf{S} -2-stacks between \mathbb{P} and \mathbb{Q} is an additive 2-functor $F : \mathbb{P} \rightarrow \mathbb{Q}$ with F an equivalence of \mathbf{S} -2-stacks. Two Picard \mathbf{S} -2-stacks are equivalent as Picard \mathbf{S} -2-stacks if there exists an equivalence of Picard \mathbf{S} -2-stacks between them.

The automorphisms $\mathcal{A}ut(e)$ of the neutral object of a Picard \mathbf{S} -2-stack form a Picard \mathbf{S} -stack. By [5§8], the homotopy groups $\pi_i(\mathbb{P})$ of a Picard \mathbf{S} -2-stack \mathbb{P} are

- $\pi_0(\mathbb{P})$ which is the sheafification of the pre-sheaf which associates, to each object U of \mathbf{S} , the group of equivalence classes of objects of $\mathbb{P}(U)$;
- $\pi_1(\mathbb{P}) = \pi_0(\mathcal{A}ut(e))$, with $\pi_0(\mathcal{A}ut(e))$ the sheafification of the pre-sheaf which associates, to each object U of \mathbf{S} , the group of isomorphism classes of objects of $\mathcal{A}ut(e)(U)$;
- $\pi_2(\mathbb{P}) = \pi_1(\mathcal{A}ut(e))$, with $\pi_1(\mathcal{A}ut(e))$ the sheaf of automorphisms of the neutral object of $\mathcal{A}ut(e)$.

The algebraic counter part of Picard \mathbf{S} -2-stacks are length 3 complexes of abelian sheaves on \mathbf{S} . In [12] the second author shows

Theorem 1.2. *It exists an equivalence of categories*

$$(1.13) \quad 2\text{st}^{\text{bb}} : \mathcal{D}^{[-2,0]}(\mathbf{S}) \longrightarrow 2\text{PICARD}^{\text{bb}}(\mathbf{S})$$

where $2\text{PICARD}^{\text{bb}}(\mathbf{S})$ is the category of Picard \mathbf{S} -2-stacks whose objects are Picard \mathbf{S} -2-stacks and whose arrows are equivalence classes of additive 2-functors.

We denote by $[]^{\text{bb}}$ the inverse equivalence of 2st^{bb} .

Example 1.3. Let \mathbb{P} and \mathbb{Q} be two Picard \mathbf{S} -2-stacks. Denote by $\text{Hom}_{2\text{PICARD}(\mathbf{S})}(\mathbb{P}, \mathbb{Q})$ the Picard \mathbf{S} -2-stack such that for any object U of \mathbf{S} , the objects of the 2-category $\text{Hom}_{2\text{PICARD}(\mathbf{S})}(\mathbb{P}, \mathbb{Q})(U)$ are additive 2-functors from $\mathbb{P}(U)$ to $\mathbb{Q}(U)$, its 1-arrows are morphisms of additive 2-functors and its 2-arrows are modifications of morphisms of additive 2-functors. By [12§4], in the derived category $\mathcal{D}(\mathbf{S})$ we have the equality

$$[\text{Hom}_{2\text{PICARD}(\mathbf{S})}(\mathbb{P}, \mathbb{Q})]^{\text{bb}} = \tau_{\leq 0} \text{RHom}([\mathbb{P}], [\mathbb{Q}]).$$

With these notation, the hom-2-groupoid of two objects \mathbb{P} and \mathbb{Q} of the 3-category $2\mathcal{PICARD}(\mathbf{S})$ is just $\mathcal{H}om_{2\mathcal{PICARD}(\mathbf{S})}(\mathbb{P}, \mathbb{Q})$.

2. THE 3-CATEGORY $\mathcal{TORS}(\mathbb{G})$ OF \mathbb{G} -TORSORS

Let \mathbf{S} be a site. We start explaining how we generalize from \mathbf{S} -stacks to \mathbf{S} -2-stacks the notions of (left/right) torsors, morphisms of torsors, 2-morphisms of torsors that have been described by the first author in [2§2]. The notion of torsor under a gr- \mathbf{S} -2-stack will be given by a collection (object, 1-arrow, 2-arrow, 3-arrow) involving objects, 1-arrows, 2-arrows and 3-arrows: the \mathbf{S} -stack in groupoids, which is the object, will be replaced by an \mathbf{S} -2-stack in 2-groupoids, the data involving 1-arrows and 2-arrows remain the same, but the coherence axioms or the compatibility conditions, that 2-arrows have to satisfy and that are given via equations of 1-arrows, are replaced by 3-arrows which express the obstruction to the above coherence axioms or compatibility conditions for 2-arrows, and moreover we require that these 3-arrows satisfies some coherence axioms or compatibility conditions that are given via equations of 2-arrows. A morphism of torsors between (object, 1-arrow, 2-arrow, 3-arrow) and (object', 1-arrow', 2-arrow', 3-arrow') is given by a collection (1-Arrow, 2-Arrow, 3-Arrow) such that

- 1-Arrow: object \rightarrow object',
- 2-Arrow expresses the compatibility between 1-Arrow, 1-arrow and 1-arrow',
- 3-Arrow expresses the obstruction to the compatibility between 2-Arrow, 2-arrow and 2-arrow'. Moreover we require the compatibility between 3-Arrow, 3-arrow and 3-arrow'.

2-morphisms and 3-morphisms of torsors are given respectively by collections (2-Arrow, 3-Arrow) and (3-Arrow) that satisfy the same conditions as before: 2-Arrow expresses the compatibility between 1-arrows, 3-Arrow expresses the obstruction to the compatibility between 2-arrows and we require the compatibility for 3-arrows.

As in Section 1, in the following definitions U will denote an object of \mathbf{S} and in the diagrams involving 2-arrows, we will put the symbol \cong in the cells which commute up to a modification of \mathbf{S} -2-stacks coming from the group like structure.

Let $\mathbb{G} = (\mathbb{G}, \otimes, \mathbf{a}, \pi)$ be a gr- \mathbf{S} -2-stack. For simplicity instead of $g_1 \otimes g_2$ we will write just $g_1 g_2$ for all $g_1, g_2 \in \mathbb{G}(U)$. As discussed in Remark 1.1, the equivalences of \mathbf{S} -2-stacks $g \otimes - : \mathbb{G} \rightarrow \mathbb{G}$ and $- \otimes g : \mathbb{G} \rightarrow \mathbb{G}$ imply that any gr- \mathbf{S} -2-stack admits a global neutral object $1_{\mathbb{G}}$ (denoted simply by 1) endowed with two natural 2-transformations of \mathbf{S} -2-stacks $\mathbf{l} : e \otimes - \Rightarrow \text{id}$ and $\mathbf{r} : - \otimes e \Rightarrow \text{id}$ which express the left and the right unit constraints and which satisfy some higher compatibility conditions (see [10]).

Definition 2.1. A *right \mathbb{G} -torsor* is given by a collection $\mathbb{P} = (\mathbb{P}, m, \mu, \Theta)$ where

- \mathbb{P} is an \mathbf{S} -2-stack in 2-groupoids;
- $m : \mathbb{P} \times \mathbb{G} \rightarrow \mathbb{P}$ is a morphism of \mathbf{S} -2-stacks, called the *action of \mathbb{G} on \mathbb{P}* . For simplicity instead of $m(p, g)$ we write just $p.g$ for any $(p, g) \in \mathbb{P} \times \mathbb{G}(U)$;
- $\mu : m \circ (\text{id}_{\mathbb{P}} \times \otimes) \Rightarrow m \circ (m \times \text{id}_{\mathbb{G}})$ is a natural 2-transformation of \mathbf{S} -2-stacks whose component at $(p, g_1, g_2) \in \mathbb{P} \times \mathbb{G}^2(U)$ is the 1-arrow $\mu_{(p, g_1, g_2)} : p.(g_1 g_2) \rightarrow (p.g_1).g_2$ of $\mathbb{P}(U)$ and which expresses the compatibility between the group law \otimes of \mathbb{G} and the action m of \mathbb{G} on \mathbb{P} ;

- Θ is a modification of **S**-2-stacks whose component at $(p, g_1, g_2, g_3) \in \mathbb{P} \times \mathbb{G}^3(U)$ is the 2-arrow

$$\begin{array}{ccc}
 & p.((g_1 g_2) g_3) & \\
 \mu_{(p, g_1 g_2, g_3)} \swarrow & & \searrow p.a_{(g_1, g_2, g_3)} \\
 (p.(g_1 g_2)).g_3 & \Leftarrow & p.(g_1 (g_2 g_3)) \\
 \mu_{(p, g_1, g_2)} g_3 \downarrow & \Theta_{(p, g_1, g_2, g_3)} & \downarrow \mu_{(p, g_1, g_2 g_3)} \\
 ((p.g_1).g_2).g_3 & \xleftarrow{\mu_{(p, g_1, g_2, g_3)}} & (p.g_1).(g_2 g_3)
 \end{array}$$

and which expresses the obstruction to the compatibility between the natural 2-transformation μ and the associativity a underlying \mathbb{G} (i.e. the obstruction to the pentagonal axiom);

such that the following conditions are satisfied:

- \mathbb{P} is locally equivalent to \mathbb{G} , i.e. $(m, \text{pr}_{\mathbb{P}}) : \mathbb{P} \times \mathbb{G} \rightarrow \mathbb{P} \times \mathbb{P}$ is an equivalence of **S**-2-stacks (here $\text{pr}_{\mathbb{P}} : \mathbb{P} \times \mathbb{G} \rightarrow \mathbb{P}$ denotes the projection to \mathbb{P});
- \mathbb{P} is locally not empty, i.e. it exists a covering sieve R of the site **S** such that for any object V of R the 2-category $\mathbb{P}(V)$ is not empty;
- the modification Θ is coherent, i.e. it satisfies the coherence axiom of Stasheff's polytope (1.5);
- the restriction of m to $\mathbb{P} \times 1_{\mathbb{G}}$ is equivalent to the identity, i.e. there exists a natural 2-transformation of **S**-2-stacks $\mathfrak{d} : m|_{(\mathbb{P} \times 1_{\mathbb{G}})} \Rightarrow \text{id}_{\mathbb{P}}$ whose component at $(p, 1_{\mathbb{G}}) \in \mathbb{P} \times 1_{\mathbb{G}}(U)$ is the 1-arrow $\mathfrak{d}_p : p.1_{\mathbb{G}} \rightarrow p$ of $\mathbb{P}(U)$. We require also the existence of two modifications of **S**-2-stacks \mathfrak{R} and \mathfrak{L} whose components at $(p, g) \in \mathbb{P} \times \mathbb{G}(U)$ are the 2-arrows

$$\begin{array}{ccc}
 p.(g1) & \xrightarrow{\mu_{(p, g, 1)}} & (p.g).1 \\
 \searrow p.\tau_g & \Leftarrow \mathfrak{R}_{(p, g)} & \swarrow \mathfrak{d}_{p.g} \\
 & p.g &
 \end{array}
 \quad
 \begin{array}{ccc}
 p.(1g) & \xrightarrow{\mu_{(p, 1, g)}} & (p.1).g \\
 \searrow p.\iota_g & \Leftarrow \mathfrak{L}_{(p, g)} & \swarrow \mathfrak{d}_{p.g} \\
 & p.g &
 \end{array}$$

(recall that ι_g and τ_g are defined as in (1.12)) and which express the obstruction to the compatibility between the restriction of m to $\mathbb{P} \times 1_{\mathbb{G}}$ and the restrictions of μ to $\mathbb{P} \times \mathbb{G} \times 1_{\mathbb{G}}$ and $\mathbb{P} \times 1_{\mathbb{G}} \times \mathbb{G}$ respectively. Moreover \mathfrak{L} and \mathfrak{R} satisfy three compatibility conditions: the first one is the compatibility between \mathfrak{L} and \mathfrak{R} , i.e. the pasting of the 2-arrows in the diagram

$$\begin{array}{ccccc}
 & p.(g_1(1g_2)) & \xrightarrow{\quad} & (p.g_1).(1g_2) & \\
 & \nearrow & & \searrow & \\
 p.((g_1 1)g_2) & \xrightarrow{\quad} & (p.(g_1 1)).g_2 & \xrightarrow{\quad} & ((p.g_1).1).g_2 \\
 & \searrow \mu_{(p, \tau_{g_1}, g_2)}^{-1} & \swarrow \mathfrak{R}_{(p, g_1).g_2} & & \\
 & p.(g_1 g_2) & \xrightarrow{\quad} & (p.g_1).g_2 &
 \end{array}$$

$\Downarrow \Theta_{(p, g_1, 1, g_2)}$

is equal to the pasting of the 2-arrows in the diagram

$$\begin{array}{ccccc}
 & p.(g_1(1g_2)) & \longrightarrow & (p.g_1).(1g_2) & \\
 & \swarrow & & \searrow & \\
 p.((g_1 1)g_2) & \xrightarrow{\quad \simeq \quad} & & & \\
 & \searrow & & \swarrow & \\
 & p.(g_1 g_2) & \longrightarrow & (p.g_1).g_2 & \\
 & \downarrow \mu_{(p.g_1, 1g_2)}^{-1} & & \downarrow \varepsilon_{(p.g_1, g_2)} & \\
 & & & & ((p.g_1).1).g_2
 \end{array}$$

The other two compatibility conditions are obtained by changing the roles of 1, g_1 and g_2 , and they express the compatibility between Θ and \mathfrak{R} , and between Θ and \mathfrak{L} . We left the explicit description of the last two conditions to the reader.

Let $\mathbb{P} = (\mathbb{P}, m_{\mathbb{P}}, \mu_{\mathbb{P}}, \Theta_{\mathbb{P}})$ and $\mathbb{Q} = (\mathbb{Q}, m_{\mathbb{Q}}, \mu_{\mathbb{Q}}, \Theta_{\mathbb{Q}})$ be two right \mathbb{G} -torsors.

Definition 2.2. A *morphism of right \mathbb{G} -torsors* from \mathbb{P} to \mathbb{Q} is given by the triplet (F, γ, Ψ) where

- $F : \mathbb{P} \rightarrow \mathbb{Q}$ is a morphism of \mathbf{S} -2-stacks;
- $\gamma : m_{\mathbb{Q}} \circ (F \times \text{id}_{\mathbb{G}}) \Rightarrow F \circ m_{\mathbb{P}}$ is a natural 2-transformation of \mathbf{S} -2-stacks whose component at $(p, g) \in \mathbb{P} \times \mathbb{G}(U)$ is the 1-arrow $\gamma_{(p, g)} : Fp.g \rightarrow F(p.g)$ (for simplicity we use the notation $.$ for both actions of \mathbb{G} on \mathbb{P} and on \mathbb{Q}) and which expresses the compatibility between the morphism of \mathbf{S} -2-stacks F and the two actions $m_{\mathbb{P}}$ and $m_{\mathbb{Q}}$ of \mathbb{G} on \mathbb{P} and on \mathbb{Q} ;
- Ψ is a modification of \mathbf{S} -2-stacks whose component at $(p, g_1, g_2) \in \mathbb{P} \times \mathbb{G}^2(U)$ is the 2-arrow

$$\begin{array}{ccc}
 & Fp.(g_1 g_2) & \\
 \gamma_{(p, g_1 g_2)} \swarrow & & \searrow \mu_{\mathbb{Q}}(Fp, g_1, g_2) \\
 F(p.(g_1 g_2)) & \xleftarrow{\quad \Psi_{(p, g_1, g_2)} \quad} & (Fp.g_1).g_2 \\
 F(\mu_{\mathbb{P}}(p, g_1, g_2)) \downarrow & & \downarrow \gamma_{(p, g_1)}.g_2 \\
 F((p.g_1).g_2) & \xleftarrow{\quad \gamma_{(p, g_1, g_2)} \quad} & F(p.g_1).g_2
 \end{array}$$

and which expresses the obstruction to the compatibility between the natural 2-transformation γ and the natural 2-transformations $\mu_{\mathbb{P}}$ and $\mu_{\mathbb{Q}}$ underlying \mathbb{P} and \mathbb{Q} . Moreover we require that the modification Ψ is compatible with the modifications $\Theta_{\mathbb{P}}$ and $\Theta_{\mathbb{Q}}$, i.e. the pasting of the 2-arrows in the diagram

$$\begin{array}{ccccccc}
 Fp.((g_1 g_2)g_3) & \longrightarrow & Fp.(g_1(g_2 g_3)) & \longrightarrow & (Fp.g_1).(g_2 g_3) & \longrightarrow & ((Fp.g_1).g_2).g_3 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \searrow \gamma_{(p, a(g_1, g_2, g_3))} & & & & & \\
 & & F(p.(g_1(g_2 g_3))) & \longrightarrow & F((p.g_1).(g_2 g_3)) & \longrightarrow & F((p.g_1).g_2).g_3 \\
 & & \downarrow F(\Theta_{\mathbb{P}}(p, g_1, g_2, g_3)) & & & & \\
 F(p.((g_1 g_2)g_3)) & \longrightarrow & F((p.(g_1 g_2)).g_3) & \longrightarrow & F(((p.g_1).g_2).g_3) & &
 \end{array}$$

is equal to the pasting of the 2-arrows in the diagram

$$\begin{array}{ccccccc}
Fp.((g_1g_2)g_3) & \longrightarrow & Fp.(g_1(g_2g_3)) & \longrightarrow & (Fp.g_1).(g_2g_3) & \longrightarrow & ((Fp.g_1).g_2).g_3 \\
\downarrow & \searrow & \downarrow & & \searrow & & \downarrow \\
& & \Theta_{\mathbb{Q}(Fp,g_1,g_2,g_3)} & & & & (F(p.g_1).g_2).g_3 \\
& & (Fp.(g_1g_2)).g_3 & & \downarrow & & \\
& & \downarrow & & \Psi_{(p,g_1,g_2).g_3} & & \\
& \swarrow & & & & & \\
& \Psi_{(p,g_1g_2,g_3)} & \Leftarrow & F(p.(g_1g_2)).g_3 & \longrightarrow & F((p.g_1).g_2).g_3 & \\
& & & \downarrow & & \downarrow & \\
& & & \gamma_{(\mu_{(p,g_1,g_2)}.g_3)} & & & \\
F(p.((g_1g_2)g_3)) & \longrightarrow & F(p.(g_1g_2)).g_3 & \longrightarrow & F((p.g_1).g_2).g_3 & &
\end{array}$$

Let (F, γ_F, Ψ_F) and (G, γ_G, Ψ_G) be two morphisms of right \mathbb{G} -torsors from \mathbb{P} to \mathbb{Q} .

Definition 2.3. A 2-morphism of right \mathbb{G} -torsors from (F, γ_F, Ψ_F) to (G, γ_G, Ψ_G) is given by the pair (α, Φ) where

- $\alpha : F \Rightarrow G$ is a natural 2-transformation of \mathbf{S} -2-stacks,
- Φ is a modification of \mathbf{S} -2-stacks whose components at $(p, g) \in \mathbb{P} \times \mathbb{G}(U)$ is the 2-arrow

$$\begin{array}{ccc}
Fp.g & \xrightarrow{\gamma_{F(p,g)}} & F(p.g) \\
\downarrow \alpha_{p.g} & \Downarrow \Phi_{(p,g)} & \downarrow \alpha_{p.g} \\
Gp.g & \xrightarrow{\gamma_{G(p,g)}} & G(p.g)
\end{array}$$

and which expresses the obstruction to the compatibility between the natural 2-transformation α and the natural 2-transformations γ_F and γ_G underlying the morphisms of right \mathbb{G} -torsors F and G . We require that the modification Φ is compatible with the modifications Ψ_F and Ψ_G , i.e. the pasting of the 2-arrows in the diagram

$$\begin{array}{ccccc}
& (Fp.g_1).g_2 & \longrightarrow & F(p.g_1).g_2 & \\
& \nearrow & & \searrow & \\
Fp.(g_1g_2) & \longrightarrow & F(p.(g_1g_2)) & \longrightarrow & F((p.g_1).g_2) \\
\downarrow & \searrow & \downarrow & & \downarrow \\
& \Phi_{(p,g_1g_2)} & & \alpha_{\mu_{(p,g_1g_2)}} & \\
Gp.(g_1g_2) & \longrightarrow & G(p.(g_1g_2)) & \longrightarrow & G((p.g_1).g_2)
\end{array}$$

is equal to the pastings of the 2-arrows in the diagram

$$\begin{array}{ccccc}
& (Fp.g_1).g_2 & \longrightarrow & F(p.g_1).g_2 & \\
& \downarrow & & \downarrow & \\
& \Phi_{(p,g_1).g_2} & & & \\
Fp.(g_1g_2) & \xrightarrow{\mu_{(\alpha_{p,g_1,g_2})}^{-1}} & (Gp.g_1).g_2 & \longrightarrow & G(p.g_1).g_2 & \longrightarrow & F((p.g_1).g_2) \\
\downarrow & \nearrow & \downarrow & & \downarrow & & \downarrow \\
& \Psi_{G(p,g_1,g_2)} & & \Phi_{(p,g_1,g_2)} & & \\
Gp.(g_1g_2) & \longrightarrow & G(p.(g_1g_2)) & \longrightarrow & G((p.g_1).g_2)
\end{array}$$

Let (α, Φ_α) and (β, Φ_β) be two 2-morphisms of right \mathbb{G} -torsors from $(F, \gamma_F, \Psi_F) : \mathbb{P} \rightarrow \mathbb{Q}$ to $(G, \gamma_G, \Psi_G) : \mathbb{P} \rightarrow \mathbb{Q}$.

Definition 2.4. A *3-morphism of right \mathbb{G} -torsors* from (α, Φ_α) to (β, Φ_β) is given by a modification of \mathbf{S} -2-stacks $\Delta : \alpha \Rightarrow \beta$ which is compatible with the modifications Φ_α and Φ_β , i.e. the following equation of 2-arrows holds

$$\begin{array}{ccc} Fp.g \longrightarrow F(p.g) & & Fp.g \longrightarrow F(p.g) \\ \downarrow & \Downarrow \Phi_{\beta(p,g)} & \downarrow \\ Gp.g \longrightarrow G(p.g) & & Gp.g \longrightarrow G(p.g) \end{array} \quad \begin{array}{c} \leftarrow \Delta_{p.g} \\ \leftarrow \Delta_{p.g} \end{array} = \begin{array}{c} \leftarrow \Delta_{p.g} \\ \leftarrow \Delta_{p.g} \end{array} \quad \begin{array}{ccc} Fp.g \longrightarrow F(p.g) & & Fp.g \longrightarrow F(p.g) \\ \downarrow & \Downarrow \Phi_{\alpha(p,g)} & \downarrow \\ Gp.g \longrightarrow G(p.g) & & Gp.g \longrightarrow G(p.g) \end{array}$$

If the gr- \mathbf{S} -2-stack \mathbb{G} acts on the left side instead of the right side, we get the definitions of left \mathbb{G} -torsor, morphism of left \mathbb{G} -torsors, 2-morphism of left \mathbb{G} -torsors and 3-morphism of left \mathbb{G} -torsors.

Definition 2.5. A \mathbb{G} -torsor $\mathbb{P} = (\mathbb{P}, m^l, m^r, \mu^l, \mu^r, \Theta^l, \Theta^r, \kappa, \Omega^r, \Omega^l)$ consists of an \mathbf{S} -2-stack in 2-groupoids \mathbb{P} endowed with a structure of left \mathbb{G} -torsor $(\mathbb{P}, m^l, \mu^l, \Theta^l)$ and with a structure of right \mathbb{G} -torsor $(\mathbb{P}, m^r, \mu^r, \Theta^r)$ which are compatible with each other. This compatibility is given by a natural 2-transformation of \mathbf{S} -2-stacks $\kappa : m^l \circ (\text{id}_{\mathbb{G}} \times m^r) \Rightarrow m^r \circ (m^l \times \text{id}_{\mathbb{G}})$ whose component at $(g_1, p, g_2) \in \mathbb{G} \times \mathbb{P} \times \mathbb{G}(U)$ is the 1-arrow $\kappa_{(g_1, p, g_2)} : g_1 \cdot (p \cdot g_2) \rightarrow (g_1 \cdot p) \cdot g_2$. We require also the existence of two modifications of \mathbf{S} -2-stacks, Ω^l whose component at $(g_1, g_2, p, g_3) \in \mathbb{G}^2 \times \mathbb{P} \times \mathbb{G}(U)$ is the 2-arrow

$$\begin{array}{ccc} (g_1 g_2) \cdot (p \cdot g_3) & \xrightarrow{\kappa_{(g_1 g_2, p, g_3)}} & ((g_1 g_2) \cdot p) \cdot g_3 \\ \mu_{(g_1, g_2, p \cdot g_3)}^l \downarrow & & \downarrow \mu_{(g_1, g_2, p) \cdot g_3}^l \\ g_1 \cdot (g_2 \cdot (p \cdot g_3)) & \xleftarrow{\Omega_{(g_1, g_2, p, g_3)}^l} & (g_1 \cdot (g_2 \cdot p)) \cdot g_3 \\ & \searrow \kappa_{(g_1, g_2, p, g_3)} & \nearrow \kappa_{(g_1, g_2, p, g_3)} \\ & g_1 \cdot ((g_2 \cdot p) \cdot g_3) & \end{array}$$

and Ω^r whose component at $(g_1, p, g_2, g_3) \in \mathbb{G} \times \mathbb{P} \times \mathbb{G}^2(U)$ is the 2-arrow

$$\begin{array}{ccc} g_1 \cdot (p \cdot (g_2 g_3)) & \xrightarrow{\kappa_{(g_1, p, g_2 g_3)}} & (g_1 \cdot p) \cdot (g_2 g_3) \\ g_1 \cdot \mu_{(p, g_2, g_3)}^r \downarrow & & \downarrow \mu_{(g_1 \cdot p, g_2, g_3)}^r \\ g_1 \cdot ((p \cdot g_2) \cdot g_3) & \xleftarrow{\Omega_{(g_1, p, g_2, g_3)}^r} & ((g_1 \cdot p) \cdot g_2) \cdot g_3 \\ & \searrow \kappa_{(g_1, p, g_2, g_3)} & \nearrow \kappa_{(g_1, p, g_2) \cdot g_3} \\ & (g_1 \cdot (p \cdot g_2)) \cdot g_3 & \end{array}$$

which express the obstruction to the compatibility between the natural 2-transformation κ and the natural 2-transformations μ^l and μ^r respectively. Moreover Ω^r and Ω^l satisfy three compatible conditions: the first one is the compatibility between Ω^r and Θ^r , i.e. up to 2-isomorphisms we have the following equation of 2-arrows

$$\Omega_{(g_1, p, g_2, g_3 g_4)}^r * \Omega_{(g_1, p, g_2, g_3, g_4)}^r * g_1 \cdot \Theta_{(p, g_2, g_3, g_4)}^r = \Theta_{(g_1 \cdot p, g_2, g_3, g_4)}^r * \Omega_{(g_1, p, g_2 g_3, g_4)}^r * \Omega_{(g_1, p, g_2, g_3)}^r \cdot g_4$$

which is a cocycle. There other two conditions, which are also cocycles, express the compatibility between Ω^l and Θ^l and between Ω^r and Ω^l .

Example 2.6. Any gr- \mathbf{S} -2-stack $\mathbb{G} = (\mathbb{G}, \otimes, \mathbf{a}, \pi)$ is endowed with a structure of left \mathbb{G} -torsor and with a structure of right \mathbb{G} -torsor: the action of \mathbb{G} on \mathbb{G} is just the group law \otimes of \mathbb{G} , the

natural 2-transformation μ is the associativity \mathbf{a} and the modification Θ is π . If moreover on \mathbb{G} we have a structure of Picard \mathbf{S} -2-stack $(\mathbb{G}, \otimes, \mathbf{a}, \pi, \mathbf{c}, \zeta, \mathfrak{h}_1, \mathfrak{h}_2, \eta)$, \mathbb{G} is in fact a \mathbb{G} -torsor: the gr- \mathbf{S} -2-stack structure underlying \mathbb{G} furnishes the structures of left and right \mathbb{G} -torsor on \mathbb{G} . The braiding \mathbf{c} implies that these two structures are compatible. We will call \mathbb{G} the *trivial \mathbb{G} -torsor*.

Let $\mathbb{P} = (\mathbb{P}, m_{\mathbb{P}}^l, m_{\mathbb{P}}^r, \mu_{\mathbb{P}}^l, \mu_{\mathbb{P}}^r, \Theta_{\mathbb{P}}^l, \Theta_{\mathbb{P}}^r, \kappa_{\mathbb{P}}, \Omega_{\mathbb{P}}^r, \Omega_{\mathbb{P}}^l)$ and $\mathbb{Q} = (\mathbb{Q}, m_{\mathbb{Q}}^l, m_{\mathbb{Q}}^r, \mu_{\mathbb{Q}}^l, \mu_{\mathbb{Q}}^r, \Theta_{\mathbb{Q}}^l, \Theta_{\mathbb{Q}}^r, \kappa_{\mathbb{Q}}, \Omega_{\mathbb{Q}}^r, \Omega_{\mathbb{Q}}^l)$ be two \mathbb{G} -torsors.

Definition 2.7. A *morphism of \mathbb{G} -torsors* from \mathbb{P} to \mathbb{Q} consists of the collection $(F, \gamma^l, \gamma^r, \Psi^l, \Psi^r, \Sigma)$ where

- $(F, \gamma^l, \Psi^l) : (\mathbb{P}, m_{\mathbb{P}}^l, \mu_{\mathbb{P}}^l, \Theta_{\mathbb{P}}^l) \rightarrow (\mathbb{Q}, m_{\mathbb{Q}}^l, \mu_{\mathbb{Q}}^l, \Theta_{\mathbb{Q}}^l)$ and $(F, \gamma^r, \Psi^r) : (\mathbb{P}, m_{\mathbb{P}}^r, \mu_{\mathbb{P}}^r, \Theta_{\mathbb{P}}^r) \rightarrow (\mathbb{Q}, m_{\mathbb{Q}}^r, \mu_{\mathbb{Q}}^r, \Theta_{\mathbb{Q}}^r)$ are morphisms of left and right \mathbb{G} -torsors respectively;
- Σ is a modification of \mathbf{S} -2-stacks whose component at $(g_1, p, g_2) \in \mathbb{G} \times \mathbb{P} \times \mathbb{G}(U)$ is the 2-arrow

$$\begin{array}{ccccc} g_1.(Fp.g_2) & \xrightarrow{g_1.\gamma_{(p,g_2)}^r} & g_1.F(p.g_2) & \xrightarrow{\gamma_{(p.g_2,g_1)}^l} & F(g_1.(p.g_2)) \\ \downarrow \kappa_{\mathbb{Q}(g_1, Fp.g_2)} & & \uparrow \Sigma_{(g_1,p,g_2)} & & \downarrow F(\kappa_{\mathbb{P}(g_1,p,g_2)}) \\ (g_1.Fp).g_2 & \xrightarrow{\gamma_{(p,g_1)}^l.g_2} & F(g_1.p).g_2 & \xrightarrow{\gamma_{(g_1.p,g_2)}^r} & F((g_1.p).g_2) \end{array}$$

which expresses the obstruction to the compatibility between the natural 2-transformations $\gamma^l, \gamma^r, \kappa_{\mathbb{P}}$ and $\kappa_{\mathbb{Q}}$. Moreover we require that the modification Σ is compatible with the modifications Ψ^l, Ψ^r, Ω^l and Ω^r . We leave the explicit description of these compatibilities to the reader.

Any morphism of \mathbb{G} -torsors $F : \mathbb{P} \rightarrow \mathbb{Q}$ is an equivalence of \mathbf{S} -2-stacks. Therefore,

Definition 2.8. Two \mathbb{G} -torsors \mathbb{P} and \mathbb{Q} are *equivalent as \mathbb{G} -torsors* if there exists a morphism of \mathbb{G} -torsors from \mathbb{P} and \mathbb{Q} .

Let $(F, \gamma_F^l, \gamma_F^r, \Psi_F^l, \Psi_F^r, \Sigma_F)$ and $(G, \gamma_G^l, \gamma_G^r, \Psi_G^l, \Psi_G^r, \Sigma_G)$ be two parallel morphisms of \mathbb{G} -torsors from \mathbb{P} to \mathbb{Q} .

Definition 2.9. A *2-morphism of \mathbb{G} -torsors* from $(F, \gamma_F^l, \gamma_F^r, \Psi_F^l, \Psi_F^r, \Sigma_F)$ to $(G, \gamma_G^l, \gamma_G^r, \Psi_G^l, \Psi_G^r, \Sigma_G)$ is given by the triplet (α, Φ^l, Φ^r) where $(\alpha, \Phi^l) : (F, \gamma_F^l, \Psi_F^l) \Rightarrow (G, \gamma_G^l, \Psi_G^l)$ and $(\alpha, \Phi^r) : (F, \gamma_F^r, \Psi_F^r) \Rightarrow (G, \gamma_G^r, \Psi_G^r)$ are 2-morphisms of left and right \mathbb{G} -torsors respectively. Moreover we require that the modifications Φ^l and Φ^r are compatible with the modifications Σ_F and Σ_G , i.e. the pasting of the 2-arrows in diagram

$$\begin{array}{ccccccc} g_1.(Fp.g_2) & \longrightarrow & (g_1.Fp).g_2 & \longrightarrow & F(g_1.p).g_2 & \longrightarrow & F((g_1.p).g_2) \\ \downarrow & \searrow & \downarrow & \Downarrow \Sigma_{F(g_1,p,g_2)} & \downarrow & \nearrow & \downarrow \\ & & g_1.F(p.g_2) & \longrightarrow & F(g_1.(p.g_2)) & & \\ & \downarrow g_1.\Phi_{(p,g_2)}^r & \downarrow & \downarrow \Phi_{(g_1.p,g_2)}^l & \downarrow & \downarrow \alpha_{\kappa_{(g_1,p,g_2)}} & \\ g_1.(Gp.g_2) & \longrightarrow & g_1.G(p.g_2) & \longrightarrow & G(g_1.p).g_2 & \longrightarrow & G((g_1.p).g_2) \end{array}$$

is equal to the pasting of the 2-arrows in diagram

$$\begin{array}{ccccccc}
g_1.(Fp.g_2) & \longrightarrow & (g_1.Fp).g_2 & \longrightarrow & F(g_1.p).g_2 & \longrightarrow & F((g_1.p).g_2) \\
\downarrow & & \downarrow \kappa_{(g_1, \alpha p, g_2)}^{-1} & & \downarrow \Phi_{(g_1, p).g_2}^l & & \downarrow \Phi_{(g_1.p).g_2}^r \\
g_1.(Gp.g_2) & \longrightarrow & (g_1.Gp).g_2 & \longrightarrow & G(g_1.p).g_2 & \longrightarrow & G((g_1.p).g_2) \\
& \searrow & & \downarrow \Sigma_{G(g_1, p, g_2)} & & \nearrow & \\
& & g_1.G(p.g_2) & \longrightarrow & G(g_1.(p.g_2)) & &
\end{array}$$

Let $(\alpha, \Phi_\alpha^l, \Phi_\alpha^r)$ and $(\beta, \Phi_\beta^l, \Phi_\beta^r)$ be two 2-morphisms of \mathbb{G} -torsors from F to G .

Definition 2.10. A 3-morphism of \mathbb{G} -torsors from $(\alpha, \Phi_\alpha^l, \Phi_\alpha^r)$ to $(\beta, \Phi_\beta^l, \Phi_\beta^r)$ is given by a modification of \mathbf{S} -2-stacks $\Delta : \alpha \Rightarrow \beta$ such that $\Delta : (\alpha, \Phi_\alpha^l) \Rightarrow (\beta, \Phi_\beta^l)$ and $\Delta : (\alpha, \Phi_\alpha^r) \Rightarrow (\beta, \Phi_\beta^r)$ are 3-morphisms of left and right \mathbb{G} -torsors respectively.

Definition-Proposition 2.11. Let \mathbb{P} and \mathbb{Q} be \mathbb{G} -torsors. Then the \mathbf{S} -2-stack $\mathrm{Hom}_{\mathrm{TORS}(\mathbb{G})}(\mathbb{P}, \mathbb{Q})$ whose

- objects are morphisms of \mathbb{G} -torsors from \mathbb{P} to \mathbb{Q} ,
- 1-arrows are 2-morphisms of \mathbb{G} -torsors,
- 2-arrows are 3-morphisms of \mathbb{G} -torsors,

is a 2-groupoid, called the 2-groupoid of morphisms of \mathbb{G} -torsors from \mathbb{P} to \mathbb{Q} .

The proof of this Proposition is left to the reader.

\mathbb{G} -torsors over \mathbf{S} form a 3-category $\mathrm{TORS}(\mathbb{G})$ where objects are \mathbb{G} -torsors and where the hom-2-groupoid of two \mathbb{G} -torsors \mathbb{P} and \mathbb{Q} is $\mathrm{Hom}_{\mathrm{TORS}(\mathbb{G})}(\mathbb{P}, \mathbb{Q})$.

We define the sum of two \mathbb{G} -torsors \mathbb{P} and \mathbb{Q} as the fibered sum (or the push-down) of \mathbb{P} and \mathbb{Q} under \mathbb{G} . In the context of torsors, the fibered sum is called the contracted product:

Definition 2.12. The contracted product $\mathbb{P} \wedge^{\mathbb{G}} \mathbb{Q}$ (or just $\mathbb{P} \wedge \mathbb{Q}$) of \mathbb{P} and \mathbb{Q} is the \mathbb{G} -torsor whose underlying \mathbf{S} -2-stack in 2-groupoids is obtained by 2-stackifying the following fibered 2-category in 2-groupoids \mathbb{D} : for any object U of \mathbf{S} ,

- (1) the objects of $\mathbb{D}(U)$ are the objects of the product $\mathbb{P} \times \mathbb{Q}(U)$, i.e. pairs (p, q) with p an object of $\mathbb{P}(U)$ and q an object of $\mathbb{Q}(U)$;
- (2) a 1-arrow $(p_1, q_1) \rightarrow (p_2, q_2)$ between two objects of $\mathbb{D}(U)$ is given by a triplet (m, g, n) where g is an object of $\mathbb{G}(U)$, $m : p_1.g \rightarrow p_2$ is a 1-arrow in $\mathbb{P}(U)$ and $n : q_1 \rightarrow g.q_2$ is a 1-arrow in $\mathbb{Q}(U)$;
- (3) a 2-arrow between two parallel 1-arrows $(m, g, n), (m', g', n') : (p_1, q_1) \rightarrow (p_2, q_2)$ of $\mathbb{D}(U)$ is given by an equivalence class of triplets (ϕ, l, θ) with $l : g \rightarrow g'$ a 1-arrow of $\mathbb{G}(U)$, $\phi : m' \circ p_1.l \Rightarrow m$ a 2-arrow of $\mathbb{P}(U)$ and $\theta : l.q_2 \circ n \Rightarrow n'$ a 2-arrow of $\mathbb{Q}(U)$. Two such triplets (ϕ, l, θ) and $(\tilde{\phi}, \tilde{l}, \tilde{\theta})$ are equivalent if there exists a 2-arrow $\gamma : l \Rightarrow \tilde{l}$ of $\mathbb{G}(U)$ such that $\tilde{\phi} * p_1.\gamma = \phi$ and $\gamma.q_2 * \tilde{\theta} = \theta$.

The contracted product of \mathbb{G} -torsors is endowed with a universal property similar to the one stated explicitly in [3, Prop 10.1].

Proposition 2.13. Let \mathbb{G} be a Picard \mathbf{S} -2-stacks. The contracted product equips the set $\mathrm{TORS}^1(\mathbb{G})$ of equivalence classes of \mathbb{G} -torsors with an abelian group law, where the neutral element is the equivalence class of the trivial \mathbb{G} -torsor \mathbb{G} , and the inverse of the equivalence class of a \mathbb{G} -torsor \mathbb{P} is the equivalence class of the $\mathrm{ad}(\mathbb{P})$ -torsor \mathbb{P} , with $\mathrm{ad}(\mathbb{P}) = \mathrm{Hom}_{\mathrm{TORS}(\mathbb{G})}(\mathbb{P}, \mathbb{P})$ (recall that \mathbb{G} and $\mathrm{ad}(\mathbb{P})$ are equivalent via $g \rightarrow (p \mapsto g.p)$).

We leave the proof of this proposition to the reader.

In order to define the notion of \mathbb{G} -torsor over an \mathbf{S} -2-stack in 2-groupoids, we need the definition of fibered product (or pull-back) for \mathbf{S} -2-stacks in 2-groupoids. Let \mathbb{P}, \mathbb{Q} and \mathbb{R} be three \mathbf{S} -2-stacks in 2-groupoids and consider two morphisms of \mathbf{S} -2-stacks $F : \mathbb{P} \rightarrow \mathbb{R}$ and $G : \mathbb{Q} \rightarrow \mathbb{R}$.

Definition 2.14. The *fibered product of \mathbb{P} and \mathbb{Q} over \mathbb{R}* is the \mathbf{S} -2-stack in 2-groupoids $\mathbb{P} \times_{\mathbb{R}} \mathbb{Q}$ defined as follows: for any object U of \mathbf{S} ,

- an object of the 2-groupoid $(\mathbb{P} \times_{\mathbb{R}} \mathbb{Q})(U)$ is a triple (p, l, q) where p is an object of $\mathbb{P}(U)$, q is an object of $\mathbb{Q}(U)$ and $l : Fp \rightarrow Gq$ is a 1-arrow in $\mathbb{R}(U)$;
- a 1-arrow $(p_1, l_1, q_1) \rightarrow (p_2, l_2, q_2)$ between two objects of $(\mathbb{P} \times_{\mathbb{R}} \mathbb{Q})(U)$ is given by the triplet (m, α, n) where $m : p_1 \rightarrow p_2$ and $n : q_1 \rightarrow q_2$ are 1-arrows in $\mathbb{P}(U)$ and $\mathbb{Q}(U)$ respectively, and $\alpha : l_2 \circ Fm \Rightarrow Gn \circ l_1$ is a 2-arrow in $\mathbb{R}(U)$;
- a 2-arrow between two parallel 1-arrows $(m, \alpha, n), (m', \alpha', n') : (p_1, l_1, q_1) \rightarrow (p_2, l_2, q_2)$ of $(\mathbb{P} \times_{\mathbb{R}} \mathbb{Q})(U)$ is given by the pair (θ, ϕ) where $\theta : m \Rightarrow m'$ and $\phi : n \Rightarrow n'$ are 2-arrows in $\mathbb{P}(U)$ and $\mathbb{Q}(U)$ respectively, satisfying the equation $\alpha' \circ (l_2 * F\theta) = (G\phi * l_1) \circ \alpha$ of 2-arrows.

The fibered product $\mathbb{P} \times_{\mathbb{R}} \mathbb{Q}$ is also called the *pull-back $F^*\mathbb{Q}$ of \mathbb{Q} via $F : \mathbb{P} \rightarrow \mathbb{R}$* or the *pull-back $G^*\mathbb{P}$ of \mathbb{P} via $G : \mathbb{Q} \rightarrow \mathbb{R}$* . It is endowed with two morphisms of \mathbf{S} -2-stacks $\text{pr}_1 : \mathbb{P} \times_{\mathbb{R}} \mathbb{Q} \rightarrow \mathbb{P}$ and $\text{pr}_2 : \mathbb{P} \times_{\mathbb{R}} \mathbb{Q} \rightarrow \mathbb{Q}$ and a natural 2-transformation of \mathbf{S} -2-stacks $\pi : G \circ \text{pr}_2 \Rightarrow F \circ \text{pr}_1$. Moreover it satisfies a universal property similar to the one stated explicitly in [3§4].

Fix an \mathbf{S} -2-stack in 2-groupoids \mathbb{Q} .

Definition 2.15. An *\mathbf{S} -2-stack in 2-groupoids over \mathbb{Q}* is a fibered product $\mathbb{A} \times_{\mathbf{0}} \mathbb{Q}$ with \mathbb{A} an \mathbf{S} -2-stack in 2-groupoids. For simplicity we denote $\mathbb{A} \times_{\mathbf{0}} \mathbb{Q}$ by $\mathbb{A}_{\mathbb{Q}}$.

Let \mathbb{G} be a gr- \mathbf{S} -2-stack and let \mathbb{Q} be an \mathbf{S} -2-stack in 2-groupoids.

Definition 2.16. A *$\mathbb{G}_{\mathbb{Q}}$ -torsor over \mathbb{Q} (or just \mathbb{G} -torsor over \mathbb{Q})* consists of an \mathbf{S} -2-stack in 2-groupoids $\mathbb{P}_{\mathbb{Q}}$ over \mathbb{Q} endowed with a structure of $\mathbb{G}_{\mathbb{Q}}$ -torsor (see Definition 2.5).

$\mathbb{G}_{\mathbb{Q}}$ -torsors over \mathbb{Q} form a 3-category, denoted $\text{TORS}(\mathbb{G}_{\mathbb{Q}})$. The explicit description of the hom-2-groupoid of two $\mathbb{G}_{\mathbb{Q}}$ -torsors over \mathbb{Q} is left to the reader.

Let \mathbb{Q} and \mathbb{R} be two \mathbf{S} -2-stacks in 2-groupoids and consider a morphism of \mathbf{S} -2-stacks $F : \mathbb{R} \rightarrow \mathbb{Q}$. If \mathbb{P} is a $\mathbb{G}_{\mathbb{Q}}$ -torsor over \mathbb{Q} , then the pull-back $F^*\mathbb{P}$ of \mathbb{P} via $F : \mathbb{R} \rightarrow \mathbb{Q}$ is a $\mathbb{G}_{\mathbb{R}}$ -torsor over \mathbb{R} . In other words, the pull-back via $F : \mathbb{R} \rightarrow \mathbb{Q}$ defines a tri-equivalence of 3-categories $F^* : \text{TORS}(\mathbb{G}_{\mathbb{Q}}) \rightarrow \text{TORS}(\mathbb{G}_{\mathbb{R}})$.

3. HOMOLOGICAL INTERPRETATION OF \mathbb{G} -TORSORS

Let \mathbb{G} be a Picard \mathbf{S} -2-stack. The complex of $\mathcal{D}^{[-2,0]}(\mathbf{S})$ corresponding to the Picard \mathbf{S} -2-stack $\mathbf{0}$, via the equivalence of categories $2\text{st}^{bb} : \mathcal{D}^{[-2,0]}(\mathbf{S}) \rightarrow 2\text{PICARD}^{bb}(\mathbf{S})$, is $\mathbf{E} = [\mathbf{e} \xrightarrow{id_{\mathbf{e}}} \mathbf{e} \xrightarrow{id_{\mathbf{e}}} \mathbf{e}]$ with \mathbf{e} the final object of the category of abelian sheaves on the site \mathbf{S} .

Lemma 3.1. *Let \mathbb{P} be a \mathbb{G} -torsor. Then the Picard \mathbf{S} -2-stack $\text{Hom}_{2\text{PICARD}(\mathbf{S})}(\mathbf{0}, \mathbb{G})$ is equivalent to $\text{Hom}_{\text{TORS}(\mathbb{G})}(\mathbb{P}, \mathbb{P})$. In particular, $\text{Hom}_{\text{TORS}(\mathbb{G})}(\mathbb{P}, \mathbb{P})$ is endowed with a Picard 2-stack structure.*

Proof. The equivalence is given by the additive 2-functor

$$\begin{aligned} \text{Hom}_{2\text{PICARD}(\mathbf{S})}(\mathbf{0}, \mathbb{G}) &\longrightarrow \text{Hom}_{\text{TORS}(\mathbb{G})}(\mathbb{P}, \mathbb{P}) \\ F &\mapsto (p \mapsto F(e).p). \end{aligned}$$

□

By the above Lemma, the homotopy groups $\pi_i(\mathrm{Hom}_{\mathrm{TORS}(\mathbb{G})}(\mathbb{P}, \mathbb{P}))$ for $i = 0, 1, 2$ are abelian groups. Since by definition $\mathrm{TORS}^{-i}(\mathbb{G}) = \pi_i(\mathrm{Hom}_{\mathrm{TORS}(\mathbb{G})}(\mathbb{P}, \mathbb{P}))$, we have

Corollary 3.2. *The sets $\mathrm{TORS}^i(\mathbb{G})$, for $i = 0, -1, -2$, are abelian groups.*

Proof of Theorem 0.1 for $i=0, -1, -2$. According to Lemma 3.1, the homotopy groups of $\mathrm{Hom}_{2\mathrm{PICARD}(\mathbf{S})}(\mathbf{0}, \mathbb{G})$ and $\mathrm{Hom}_{\mathrm{TORS}(\mathbb{G})}(\mathbb{P}, \mathbb{P})$ are isomorphic and so by Example 1.3 $\mathrm{TORS}^i(\mathbb{G}) \cong \pi_{-i}(\mathrm{Hom}_{2\mathrm{PICARD}(\mathbf{S})}(\mathbf{0}, \mathbb{G})) \simeq H^i(\tau_{\leq 0}\mathrm{RHom}(\mathbf{E}, [\mathbb{G}]^{bb}))$. Since the functor $\Gamma(-)$ of global sections is isomorphic to the functor $\mathrm{Hom}(\mathbf{E}, -)$, we can conclude that $\mathrm{TORS}^i(\mathbb{G}) \cong H^i(\tau_{\leq 0}\mathrm{R}\Gamma([\mathbb{G}]^{bb})) = H^i([\mathbb{G}]^{bb})$. \square

Before we prove Theorem 0.1 for $i = 1$, we state the following Definitions:

Definition 3.3. A \mathbb{G} -torsor \mathbb{P} is *trivial* if one of the following equivalent conditions is satisfied:

- (1) there exists a morphism of \mathbf{S} -2-stacks $\mathbf{0} \rightarrow \mathbb{P}$;
- (2) \mathbb{P} admits a global section, i.e. $\Gamma(\mathbb{P}) \neq \emptyset$;
- (3) \mathbb{P} is equivalent as \mathbb{G} -torsor (see Def. 2.8) to the neutral object \mathbb{G} of the group law defined in Definition 2.12, i.e. there exists a morphism of \mathbb{G} -torsors from \mathbb{P} and \mathbb{G} .

Definition 3.4. If \mathbb{P} is an \mathbf{S} -2-stack in 2-groupoids, the *Picard \mathbf{S} -2-stack generated by \mathbb{P}* is the Picard \mathbf{S} -2-stack $\mathbb{Z}[\mathbb{P}]$ defined as follows: for any object U of \mathbf{S} , an object of $\mathbb{Z}[\mathbb{P}](U)$ consists of a finite formal sum $\sum_i n_i [p_i]$ with $n_i \in \mathbb{Z}$ and p_i an object of $\mathbb{P}(U)$.

Proof of Theorem 0.1 for $i=1$. First we construct a morphism from the group $\mathrm{TORS}^1(\mathbb{G})$ of equivalence classes of \mathbb{G} -torsors to the group $H^1([\mathbb{G}]^{bb})$

$$\Theta: \mathrm{TORS}^1(\mathbb{G}) \longrightarrow H^1([\mathbb{G}]^{bb}),$$

and a morphism from the group $H^1([\mathbb{G}]^{bb})$ to the group $\mathrm{TORS}^1(\mathbb{G})$

$$\Psi: H^1([\mathbb{G}]^{bb}) \longrightarrow \mathrm{TORS}^1(\mathbb{G}).$$

Then we check that $\Theta \circ \Psi = \mathrm{id} = \Psi \circ \Theta$ and that Θ is an homomorphism of groups.

Before the proof we fix the following notation: if A is a complex of $\mathcal{D}^{[-2,0]}(\mathbf{S})$ we denote by \mathbb{A} the corresponding Picard \mathbf{S} -2-stack $2\mathrm{st}^{bb}(A)$. Moreover if $f: A \rightarrow B$ is a morphism in $\mathcal{D}^{[-2,0]}(\mathbf{S})$, we denote by $F: \mathbb{A} \rightarrow \mathbb{B}$ a representative of the equivalence class of additive 2-functors corresponding to the morphism f via the equivalence of categories (1.13).

(1) Construction of Θ : Let \mathbb{P} be a \mathbb{G} -torsor and let $\mathbb{Z}[\mathbb{P}]$ be the Picard \mathbf{S} -2-stack generated by \mathbb{P} . Consider the additive 2-functor

$$H: \mathbb{Z}[\mathbb{P}] \rightarrow \mathbb{Z}[\mathbf{0}]$$

which associates to an object $\sum_i n_i [p_i]$ of $\mathbb{Z}[\mathbb{P}](U)$ the object $\sum_i n_i$ of $\mathbb{Z}[\mathbf{0}](U)$, for U an object of \mathbf{S} . The homotopy kernel $\mathrm{Ker}(H)$ of H is the Picard \mathbf{S} -2-stack whose objects are sums of the form $[p] - [p']$, with p, p' objects of $\mathbb{P}(U)$. Clearly $\mathbb{Z}[\mathbb{P}]$ is an extension of Picard \mathbf{S} -2-stacks of $\mathbb{Z}[\mathbf{0}]$ by $\mathrm{Ker}(H)$ (see [3, Def 5.1])

$$\mathrm{Ker}(H) \longrightarrow \mathbb{Z}[\mathbb{P}] \longrightarrow \mathbb{Z}[\mathbf{0}].$$

Consider now the additive 2-functor $L: \mathrm{Ker}(H) \rightarrow \mathbb{G}$ which associates to an object $[p] - [p']$ of $\mathrm{Ker}(H)(U)$ the object g of $\mathbb{G}(U)$ such that $g.p = p'$. According to [3, Def 7.3], the push-down $L_*\mathbb{Z}[\mathbb{P}]$ of the extension $\mathbb{Z}[\mathbb{P}]$ via the additive 2-functor $L: \mathrm{Ker}(H) \rightarrow \mathbb{G}$ is an extension of $\mathbb{Z}[\mathbf{0}]$ by \mathbb{G}

$$\mathbb{G} \longrightarrow L_*\mathbb{Z}[\mathbb{P}] \longrightarrow \mathbb{Z}[\mathbf{0}].$$

By [3, Prop. 6.7, Rem. 6.6] to this extension $L_*\mathbb{Z}[\mathbb{P}]$ of Picard **S**-2-stacks is associated the distinguished triangle $[\mathbb{G}]^{bb} \rightarrow [L_*\mathbb{Z}[\mathbb{P}]]^{bb} \rightarrow \mathbf{E} \rightarrow +$ in $\mathcal{D}(\mathbf{S})$ which furnishes the long exact sequence

$$\cdots \longrightarrow H^0([\mathbb{G}]^{bb}) \longrightarrow H^0([L_*\mathbb{Z}[\mathbb{P}]]^{bb}) \longrightarrow H^0(\mathbf{E}) \xrightarrow{\partial} H^1([\mathbb{G}]^{bb}) \longrightarrow \cdots$$

We set $\Theta(\mathbb{P}) = \partial(1)$, where the element 1 of $H^0(\mathbf{E})$ corresponds to the global neutral object $e \in \Gamma(\mathbf{0})$ of the Picard **S**-2-stack $\mathbf{0}$. The naturality of the connecting map ∂ implies that $\Theta(\mathbb{P})$ depends only on the equivalence class of the \mathbb{G} -torsor \mathbb{P} .

(2) Construction of Ψ : Let G be the complex $[\mathbb{G}]^{bb}$ of $\mathcal{D}^{[-2,0]}(\mathbf{S})$ corresponding to the Picard **S**-2-stack \mathbb{G} . Choose a complex $I = [I^{-2} \rightarrow I^{-1} \rightarrow I^0]$ of $\mathcal{D}^{[-2,0]}(\mathbf{S})$ such that I^{-2}, I^{-1}, I^0 are injective and such that there exists an injective morphism of complexes $s: G \rightarrow I$. We complete s into a distinguished triangle $G \xrightarrow{s} I \xrightarrow{t} \text{MC}(s) \rightarrow +$ in $\mathcal{D}(\mathbf{S})$. Setting $K = \tau_{\geq -2}\text{MC}(s)$, the above distinguished triangle furnishes an extension of Picard **S**-2-stacks

$$\mathbb{G} \xrightarrow{S} \mathbb{I} \xrightarrow{T} \mathbb{K},$$

and the long exact sequence

$$(3.1) \quad \cdots \longrightarrow H^0(G) \longrightarrow H^0(I) \xrightarrow{to} H^0(K) \xrightarrow{\partial} H^1(G) \longrightarrow 0.$$

Given an element x of $H^1(G)$, choose an element u of $H^0(K)$ such that $\partial(u) = x$. Remark that via the equivalence of categories 2st^{bb} (1.13), the element $u \in H^0(K)$ corresponds to a global section $U \in \Gamma(\mathbb{K})$ of \mathbb{K} , i.e. to an additive 2-functor $U: \mathbf{0} \rightarrow \mathbb{K}$. Using the notion of pull-back (or fibered product) of **S**-2-stacks in 2-groupoids given in Definition 2.14, consider the pull-back $U^*\mathbb{I}$ of \mathbb{I} via $U: \mathbf{0} \rightarrow \mathbb{K}$. This pull-back $U^*\mathbb{I}$, which is an **S**-2-stack in 2-groupoids not necessarily endowed with a Picard **S**-2-stack structure, is a \mathbb{G} -torsor: in fact, the objects of $U^*\mathbb{I}$ are the objects of \mathbb{I} whose image via $T: \mathbb{I} \rightarrow \mathbb{K}$ is $U(e)$ (recall that e is the only object of $\mathbf{0}$) and the action $\mathbb{G} \times U^*\mathbb{I} \rightarrow U^*\mathbb{I}$ of \mathbb{G} on $U^*\mathbb{I}$ is given by $(g, i) \mapsto S(g).i$, where g is an object of \mathbb{G} , i is an object of \mathbb{I} such that $T(i) = U(e)$, and "." is the group law of the Picard **S**-2-stack \mathbb{I} .

We set $\Psi(x) = U^*\mathbb{I}$ i.e. to be precise $\Psi(x)$ is the equivalence class of the \mathbb{G} -torsor $U^*\mathbb{I}$. Now we check that the morphism Ψ is well defined, i.e. $\Psi(u)$ doesn't depend on the lift u of x . Let $u' \in H^0(K)$ be another lift of x . From the exactness of the sequence (3.1), there exists $f \in H^0(I)$ such that $u' - u = t \circ f$, i.e. we have the following commutative diagram

$$\begin{array}{ccc} \mathbf{0} & \xrightarrow{\text{id}_{\mathbf{0}}} & \mathbf{0} \\ F \downarrow & \nearrow \pi & \downarrow U' - U \\ \mathbb{K} & \xrightarrow{T} & \mathbb{L} \end{array}$$

Consider now the pull-back $(U' - U)^*\mathbb{I}$ of \mathbb{I} via $U' - U: \mathbf{0} \rightarrow \mathbb{K}$ according to Definition 2.14. The universal property of this pull-back $(U' - U)^*\mathbb{I}$ applied to the above diagram furnishes a morphism of **S**-2-stacks $H: \mathbf{0} \rightarrow (U' - U)^*\mathbb{I}$, i.e. the \mathbb{G} -torsor $(U' - U)^*\mathbb{I}$ admits a global section H . Hence by Definition 3.3 the \mathbb{G} -torsor $(U' - U)^*\mathbb{I}$ is trivial, which means that the \mathbb{G} -torsors $U'^*\mathbb{I}$ and $U^*\mathbb{I}$ are equivalent.

(3) and (4) The proofs of the equalities $\Theta \circ \Psi = \text{id}$ and $\Psi \circ \Theta = \text{id}$, which imply that Θ is surjective and injective, are similar to the steps (3) and (4) of the proof of [3, Thm 1.1 $i = 1$]. Therefore we left them to the reader.

(5) Θ is a group homomorphism: Consider two \mathbb{G} -torsors \mathbb{P}, \mathbb{P}' . With the notations of (2) we can suppose that $\mathbb{P} = U^*\mathbb{I}$ and $\mathbb{P}' = U'^*\mathbb{I}$ with $U, U': \mathbf{0} \rightarrow \mathbb{K}$ two additive 2-functors corresponding to the elements $u, u' \in H^0(K)$. The sum in $\text{Tors}^1(\mathbb{G})$ is defined as the

contracted product of two torsors which is a fibered sum. Since fibered sums and fibered products commute, we have the following equalities

$$\mathbb{P} + \mathbb{P}' = U^*\mathbb{I} \wedge^{\mathbb{G}} U'^*\mathbb{I} = (U + U')^*(\mathbb{I} +^{\mathbb{G}} \mathbb{I}) = (U + U')^*(\mathbb{I})$$

where $\mathbb{I} +^{\mathbb{G}} \mathbb{I}$ denotes the fibered sum of \mathbb{I} with itself under \mathbb{G} (see [3, Def 4.8]). These equalities shows that $\Theta(\mathbb{P} + \mathbb{P}') = \partial(u + u')$ where $\partial: H^0(K) \rightarrow H^1(G)$ is the connecting map of the long exact sequence (3.1). Hence, $\Theta(\mathbb{P} + \mathbb{P}') = \partial(u + u') = \partial(u) + \partial(u') = \Theta(\mathbb{P}) + \Theta(\mathbb{P}')$. \square

Proof of Corollary 0.2. Let G be the complex $[\mathbb{G}]^{bb}$ of $\mathcal{D}^{[-2,0]}(\mathbf{S})$. If $Q = [Q^{-2} \rightarrow Q^{-1} \rightarrow Q^0]$ is the complex of sheaves of groups corresponding to \mathbb{Q} , the complex of abelian sheaves corresponding to $\mathbb{Z}[\mathbb{Q}]$ is $\mathbb{Z}[Q] = [\mathbb{Z}[Q^{-2}] \rightarrow \mathbb{Z}[Q^{-1}] \rightarrow \mathbb{Z}[Q^0]]$, with $\mathbb{Z}[Q^i]$ the abelian sheaf generated by Q^i according to [8] Exposé IV 11. By definition of $\mathbb{Z}[Q]$, the functor

$$G \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[Q], G)$$

is isomorphic to the functor

$$G \longrightarrow G(Q) = H^0(Q, G_Q),$$

with $G_Q = [\mathbb{G}_Q]^{bb}$ by [3, Prop 4.4]. Taking the respective derived functors, we get the isomorphisms $\mathrm{Ext}^i(\mathbb{Z}[Q], G) \cong H^i(Q, G_Q)$ for $i = -2, -1, 0, 1$, and so we can conclude using the homological interpretation of torsors furnished by Theorem 0.1 and the homological interpretation of extensions of Picard \mathbf{S} -2stacks given by [3, Thm 1.1]. \square

4. DESCRIPTION OF EXTENSIONS OF PICARD 2-STACKS IN TERMS OF TORSORS

Let \mathbb{P} and \mathbb{G} be two Picard \mathbf{S} -2stacks. If K is a subset of a finite set E , $p_K: \mathbb{P}^E \rightarrow \mathbb{P}^K$ is the projection to the factors belonging to K , and $\otimes_K: \mathbb{P}^E \rightarrow \mathbb{P}^{E-K+1}$ is the group law $\otimes: \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ on the factors belonging to K . If ι is a permutation of the set E , $\mathrm{Perm}(\iota): \mathbb{P}^E \rightarrow \mathbb{P}^{\iota(E)}$ is the permutation of the factors according to ι . Moreover let $\mathbf{s}: \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P} \times \mathbb{P}$ be the morphism of \mathbf{S} -2stacks that exchanges the factors and let $D: \mathbb{P} \rightarrow \mathbb{P} \times \mathbb{P}$ be the diagonal morphism of \mathbf{S} -2stacks.

Proposition 4.1. *To have an extension $\mathbb{E} = (\mathbb{E}, I, J)$ of \mathbb{P} by \mathbb{G} is equivalent to have*

- (1) *a $\mathbb{G}_{\mathbb{P}}$ -torsor \mathbb{E} over \mathbb{P} ;*
- (2) *a morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors $M: p_1^* \mathbb{E} \wedge p_2^* \mathbb{E} \rightarrow \otimes^* \mathbb{E}$. Here $\otimes^* \mathbb{E}$ is the pull-back of \mathbb{E} via the group law $\otimes: \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ of \mathbb{P} and for $i = 1, 2$, $p_i^* \mathbb{E}$ is the pull-back of \mathbb{E} via the i -th projection $p_i: \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ (these pull-backs are pull-backs of \mathbf{S} -2stacks in 2-groupoids according to Definition 2.14);*
- (3) *a 2-morphism of $\mathbb{G}_{\mathbb{P}^3}$ -torsors $\alpha: M \circ (M \wedge \mathrm{id}) \Rightarrow M \circ (\mathrm{id} \wedge M)$;*
- (4) *a 3-morphism of $\mathbb{G}_{\mathbb{P}^4}$ -torsors*

$$\mathbf{a}: p_{234}^* \alpha \circ \otimes_{23}^* \alpha \circ p_{123}^* \alpha \Rightarrow \otimes_{34}^* \alpha \circ \otimes_{12}^* \alpha$$

whose pull-back over \mathbb{P}^5 satisfies the equality

$$(4.1) \quad \otimes_{45}^* \mathbf{a} \circ \otimes_{23}^* \mathbf{a} \circ p_{2345}^* \mathbf{a} = \otimes_{12}^* \mathbf{a} \circ p_{1234}^* \mathbf{a} \circ \otimes_{34}^* \mathbf{a}.$$

- (5) *a 2-morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors $\chi: M \Rightarrow M \circ \mathbf{s}$;*
- (6) *a 3-morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors $\mathbf{s}: \chi \circ \chi \Rightarrow \mathrm{id}$ satisfying the equation of 2-arrows obtained from (1.6) by replacing \mathbf{c} with χ and ζ with \mathbf{s} ;*
- (7) *two 3-morphisms of $\mathbb{G}_{\mathbb{P}^3}$ -torsors*

$$\mathbf{c}_1: \mathrm{Perm}(132)^* \alpha \circ \otimes_{23}^* \chi \circ \alpha \Rightarrow p_{13}^* \chi \circ \mathrm{Perm}(12)^* \alpha \circ p_{12}^* \chi$$

$$\mathbf{c}_2: \mathrm{Perm}(123)^* \alpha^{-1} \circ \otimes_{12}^* \mathbf{s}^* \chi^{-1} \circ \alpha^{-1} \Rightarrow p_{13}^* \mathbf{s}^* \chi^{-1} \circ \mathrm{Perm}(23)^* \alpha^{-1} \circ p_{23}^* \mathbf{s}^* \chi^{-1}$$

which satisfy the compatibility conditions obtained from (1.8), (1.9), (1.10) by replacing ζ with \mathfrak{s} , \mathfrak{h}_i with \mathfrak{c}_i for $i = 1, 2$, and whose pull-back over \mathbb{P}^4 satisfy

$$(4.2) \quad \text{Perm}(12)^* \mathfrak{a} \circ p_{134}^* \mathfrak{c}_1 \circ \otimes_{34}^* \mathfrak{c}_1 \circ \mathfrak{a} = p_{123}^* \mathfrak{c}_1 \circ \text{Perm}(132)^* \mathfrak{a} \circ \text{Perm}(1432)^* \mathfrak{a} \circ \otimes_{23}^* \mathfrak{c}_1.$$

- (8) a 3-morphism of $\mathbb{G}_{\mathbb{P}}$ -torsors $\mathfrak{p} : D^* \chi \Rightarrow \text{id}$ satisfying $\mathfrak{p} * \mathfrak{p} = \mathfrak{s}$ and the compatibility condition obtained from (1.11) by replacing π with \mathfrak{a} , ζ with \mathfrak{s} , \mathfrak{h}_i with \mathfrak{c}_i for $i = 1, 2$, η with \mathfrak{p} .

Proof. (I) Let $\mathbb{E} = (\mathbb{E}, I, J)$ be an extension of \mathbb{P} by \mathbb{G} . Via the additive 2-functor $I : \mathbb{G} \rightarrow \mathbb{E}$, the Picard \mathbf{S} -2-stack \mathbb{G} acts on the left side and on the right side of \mathbb{E} furnishing a structure of \mathbb{G} -torsor to \mathbb{E} . Since the additive 2-functor $J : \mathbb{E} \rightarrow \mathbb{P}$ induces a surjection $\pi_0(J) : \pi_0(\mathbb{E}) \rightarrow \pi_0(\mathbb{P})$ on the π_0 , \mathbb{E} is in fact a $\mathbb{G}_{\mathbb{P}}$ -torsor over \mathbb{P} (1). The group law $\otimes : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{E}$ of \mathbb{E} furnishes a morphism of \mathbf{S} -2-stacks $p_1^* \mathbb{E} \times p_2^* \mathbb{E} \rightarrow \otimes^* \mathbb{E}$ over $\mathbb{P} \times \mathbb{P}$. The existence for any $g \in \mathbb{G}$ and $a, b \in \mathbb{E}$ of the associativity constraint $\mathfrak{a}_{(a,g,b)} : (ag)b \rightarrow a(gb)$ implies that this morphism of \mathbf{S} -2-stacks $p_1^* \mathbb{E} \times p_2^* \mathbb{E} \rightarrow \otimes^* \mathbb{E}$ factorizes via the contracted product $M : p_1^* \mathbb{E} \wedge p_2^* \mathbb{E} \rightarrow \otimes^* \mathbb{E}$. The existence for any $g \in \mathbb{G}$ and $a, b \in \mathbb{E}$ of the associativity constraints $\mathfrak{a}_{(g,a,b)} : (ga)b \rightarrow g(ab)$ and $\mathfrak{a}_{(a,b,g)} : (ab)g \rightarrow a(bg)$ implies that the morphism of \mathbf{S} -2-stacks $M : p_1^* \mathbb{E} \wedge p_2^* \mathbb{E} \rightarrow \otimes^* \mathbb{E}$ is in fact a morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors once we consider on $p_1^* \mathbb{E} \wedge p_2^* \mathbb{E}$ the following structure of $\mathbb{G}_{\mathbb{P}^2}$ -torsors: the left (resp. right) action of $\mathbb{G}_{\mathbb{P}^2}$ on $p_1^* \mathbb{E} \wedge p_2^* \mathbb{E}$ comes from the left (resp. right) action of $\mathbb{G}_{\mathbb{P}^2}$ on $p_1^* \mathbb{E}$ (resp. $p_2^* \mathbb{E}$) (2). Now the associativity $\mathfrak{a} : \otimes \circ (\otimes \times \text{id}_{\mathbb{E}}) \Rightarrow \otimes \circ (\text{id}_{\mathbb{E}} \times \otimes)$ implies the 2-morphism of $\mathbb{G}_{\mathbb{P}^3}$ -torsors $\alpha : M \circ (M \wedge \text{id}) \Rightarrow M \circ (\text{id} \wedge M)$ over $\mathbb{P} \times \mathbb{P} \times \mathbb{P}$ (3). The modification π of \mathbf{S} -2-stacks (1.1), expressing the obstruction to the coherence of the associativity \mathfrak{a} (i.e. the obstruction to the pentagonal axiom) and satisfying the coherence axiom of Stasheff's polytope (1.5), is equivalent to the 3-morphism of $\mathbb{G}_{\mathbb{P}^4}$ -torsors \mathfrak{a} satisfying the equality (4.1) (4). The braiding $\mathfrak{c} : \otimes \circ \mathfrak{s} \Rightarrow \otimes$ furnishes the 2-morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors $\chi : M \Rightarrow M \circ \mathfrak{s}$ over $\mathbb{P} \times \mathbb{P}$. The modification ζ of \mathbf{S} -2-stacks (1.2), expressing the obstruction to the coherence of the braiding \mathfrak{c} and satisfying the coherence condition (1.6), is equivalent to the 3-morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors \mathfrak{s} with its coherence condition (6). The modifications \mathfrak{h}_1 and \mathfrak{h}_2 of \mathbf{S} -2-stacks (1.3), expressing the obstruction to the compatibility between the associativity \mathfrak{a} and the braiding \mathfrak{c} (i.e. the obstruction to the hexagonal axiom) and satisfying the compatibility conditions (1.7), (1.8), (1.9), (1.10), are equivalent to the 3-morphisms of $\mathbb{G}_{\mathbb{P}^3}$ -torsors \mathfrak{c}_1 and \mathfrak{c}_2 with their compatibility conditions (7). Remark that the condition (1.7) is equivalent to the equality (4.2). Finally, the modification η of \mathbf{S} -2-stacks (1.4), expressing the obstruction to the strictness of the braiding \mathfrak{c} and satisfying $\eta * \eta = \zeta$ and the compatibility condition (1.11), is equivalent to the 3-morphism of $\mathbb{G}_{\mathbb{P}}$ -torsors \mathfrak{p} with its compatibility conditions (8).

(II) Now suppose we have the data $(\mathbb{E}, M, \alpha, \mathfrak{a}, \chi, \mathfrak{s}, \mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{p})$ given in (1)-(6). We will show that the $\mathbb{G}_{\mathbb{P}}$ -torsor \mathbb{E} over \mathbb{P} is a Picard \mathbf{S} -2-stack endowed with a structure of extension of \mathbb{P} by \mathbb{G} . The morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors $M : p_1^* \mathbb{E} \wedge p_2^* \mathbb{E} \rightarrow \otimes^* \mathbb{E}$ over $\mathbb{P} \times \mathbb{P}$ defines a group law $\otimes : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{E}$ on the \mathbf{S} -2-stack of 2-groupoids \mathbb{E} . The data α and χ furnish the associativity $\mathfrak{a} : \otimes \circ (\otimes \times \text{id}_{\mathbb{E}}) \Rightarrow \otimes \circ (\text{id}_{\mathbb{E}} \times \otimes)$ and the braiding $\mathfrak{c} : \otimes \circ \mathfrak{s} \Rightarrow \otimes$ which express respectively the associativity and the commutativity constraints of the group law \otimes of \mathbb{E} . As already observed in (I), the data $\mathfrak{a}, \mathfrak{s}, \mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{p}$ give respectively the modifications of \mathbf{S} -2-stacks π (1.1), ζ (1.2), $\mathfrak{h}_1, \mathfrak{h}_2$ (1.3), η (1.4), with their coherence and compatibility conditions. Since any morphism of \mathbb{G} -torsors is an equivalence of \mathbf{S} -2-stacks, the morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors $M : p_1^* \mathbb{E} \wedge p_2^* \mathbb{E} \rightarrow \otimes^* \mathbb{E}$ implies that for any object $a \in \mathbb{E}$, the left multiplication by a , $a \otimes - : \mathbb{E} \rightarrow \mathbb{E}$, is an equivalence of \mathbf{S} -2-stacks (recall that by Remark 1.1 this property of the left multiplication to be an equivalence implies that \mathbb{E} admits a global neutral object e and that any object of \mathbb{E} admits an inverse). Hence, \mathbb{E} is a Picard \mathbf{S} -2-stack.

If $J : \mathbb{E} \rightarrow \mathbb{P}$ denotes the morphism of \mathbf{S} -2-stacks underlying the structure of $\mathbb{G}_{\mathbb{P}}$ -torsor over \mathbb{P} , J must be a surjection on the isomorphism classes of objects, i.e. $\pi_0(J) : \pi_0(\mathbb{E}) \rightarrow \pi_0(\mathbb{P})$ is surjective. Moreover the compatibility of J with the morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors $M : p_1^* \mathbb{E} \wedge p_2^* \mathbb{E} \rightarrow \otimes^* \mathbb{E}$ over $\mathbb{P} \times \mathbb{P}$ implies that J is an additive 2-functor. The neutral object of the Picard \mathbf{S} -2-stack \mathbb{E} induces a morphism of \mathbb{G} -torsors from \mathbb{G} to the pull-back $\mathbf{0}^* \mathbb{E}$ of \mathbb{E} via the additive 2-functor $\mathbf{0} : \mathbf{0} \rightarrow \mathbb{P}$, i.e. $\mathbf{0}^* \mathbb{E}$ is equivalent as \mathbb{G} -torsor to \mathbb{G} . Recalling that by [3, Def 4.9] the homotopy kernel $\mathbb{Ker}(J)$ of J is the pull-back $\mathbf{0}^* \mathbb{E}$, let I be the composite $\mathbb{G} \cong \mathbf{0}^* \mathbb{E} = \mathbb{Ker}(J) \rightarrow \mathbb{E}$ where $\mathbf{0}^* \mathbb{E} = \mathbb{E} \times_{\mathbb{P}} \mathbf{0} \rightarrow \mathbb{E}$ is the additive 2-functor underlying the fibered product $\mathbb{E} \times_{\mathbb{P}} \mathbf{0}$. Clearly I is an additive 2-functor. We can conclude that (\mathbb{E}, I, J) is an extension of \mathbb{P} by \mathbb{G} . \square

As a consequence of this Proposition we get the proof of Theorem 0.3, whose details are left to the reader.

Remark 4.2. The above Proposition generalizes to Picard \mathbf{S} -2-stacks the results [4, Thm 3.2.2] and [2, Thm 4.1] where a similar statement is proved for gr- \mathbf{S} -stacks and for Picard \mathbf{S} -stacks respectively. Thanks to Remark 1.1 our proof is simpler respect to the ones of Breen and Bertolin, since it does not involve the global neutral object e of \mathbb{E} and the inverse objects of objects of \mathbb{E} .

5. RIGHT RESOLUTION OF $\mathcal{E}xt(\mathbb{P}, \mathbb{G})$

In this section we prove Corollary 0.4, constructing a 3-category $\Psi_{\mathbb{L}(\mathbb{P})}(\mathbb{G})$, which will be tri-equivalent to the 3-category $\mathcal{E}xt(\mathbb{P}, \mathbb{G})$ of extensions of Picard \mathbf{S} -2-stacks of \mathbb{P} by \mathbb{G} , and a canonical flat partial resolution $\mathbb{L}(\mathbb{P})$ of the Picard \mathbf{S} -2-stack \mathbb{P} .

A cochain complex of Picard \mathbf{S} -2-stacks $\dots \rightarrow \mathbb{L}^{-1} \xrightarrow{D^{-1}} \mathbb{L}^0 \xrightarrow{D^0} \mathbb{L}^1 \xrightarrow{D^1} \dots$ consists of

- Picard \mathbf{S} -2-stacks \mathbb{L}^i for $i \in \mathbb{Z}$;
- additive 2-functors $D^i : \mathbb{L}^i \rightarrow \mathbb{L}^{i+1}$ for $i \in \mathbb{Z}$;
- morphisms of additive 2-functors $\partial^i : D^{i+1} \circ D^i \Rightarrow 0$ for $i \in \mathbb{Z}$;
- modifications of morphisms of additive 2-functors

$$(5.1) \quad \begin{array}{ccccc} (D^{i+2} D^{i+1}) D^i & \xrightarrow{a} & D^{i+2} (D^{i+1} D^i) & \xrightarrow{D^{i+2} * \partial^i} & D^{i+2} 0 \\ \partial^{i+1} * D^i \Downarrow & & \Downarrow \Delta_{(i+2, i+1, i)} & & \Downarrow \\ 0 D^i & \xrightarrow{\quad\quad\quad} & & & 0 \end{array}$$

which satisfy the following equation of modifications: the pasting of the modifications

$$\begin{array}{ccccccc} & & (D^{i+3} (D^{i+2} D^{i+1})) D^i & \xRightarrow{\quad} & D^{i+3} ((D^{i+2} D^{i+1}) D^i) & \xRightarrow{\quad} & D^{i+3} (D^{i+2} (D^{i+1} D^i)) \\ & \nearrow & \Downarrow & & \Downarrow & & \Downarrow \\ & & \Downarrow \Delta_{(i+3, i+2, i+1)} * D^i & & \Downarrow \circ_{(a, \partial^{i+1})} & & \Downarrow \\ ((D^{i+3} D^{i+2}) D^{i+1}) D^i & \xRightarrow{\quad} & (D^{i+3} 0) D^i & \xRightarrow{\quad} & D^{i+3} (0 D^i) & \xRightarrow{\quad} & D^{i+3} (D^{i+2} 0) \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ (0 D^{i+1}) D^i & \xRightarrow{\quad} & 0 D^i & \xRightarrow{\quad} & 0 & \xRightarrow{\quad} & D^{i+3} 0 \end{array}$$

is equal to the pasting of the modifications in the diagram below

$$\begin{array}{ccccc}
& ((D^{i+3}D^{i+2})D^{i+1})D^i & \Longrightarrow & (D^{i+3}(D^{i+2}D^{i+1}))D^i & \Longrightarrow & D^{i+3}((D^{i+2}D^{i+1})D^i) \\
& \swarrow & & \searrow & & \Downarrow \\
(0D^{i+1})D^i & & \xLeftarrow{\circ_{(a,\partial^{i+2})}} & & (D^{i+3}D^{i+2})(D^{i+1}D^i) & \xRightarrow{\pi_{(i+3,i+2,i+1,i)}} & D^{i+3}(D^{i+2}(D^{i+1}D^i)) \\
& \searrow & & \swarrow & & \Downarrow \\
0D^i & \xrightarrow{\simeq} & 0(D^{i+1}D^i) & \xLeftarrow{\circ_{(\partial^{i+2},\partial^i)}} & (D^{i+3}D^{i+2})0 & \xRightarrow{\circ_{(a,\partial^i)}} & D^{i+3}(D^{i+2}0) \\
& \searrow & \Downarrow & \swarrow & & \Downarrow \\
& & 0 & \xrightarrow{\simeq} & & D^{i+3}0
\end{array}$$

Let \mathbb{G} be a Picard \mathbf{S} -2-stack and let $\mathbb{L} : 0 \rightarrow \mathbb{T} \xrightarrow{D^{\mathbb{T}}} \mathbb{S} \xrightarrow{D^{\mathbb{S}}} \mathbb{R} \xrightarrow{D^{\mathbb{R}}} \mathbb{Q} \xrightarrow{D^{\mathbb{Q}}} \mathbb{P} \rightarrow 0$ be a complex of Picard \mathbf{S} -2-stacks with $\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S}$, and \mathbb{T} in degrees 0, -1, -2, -3 and -4, respectively. To the complex \mathbb{L} and to \mathbb{G} , we associate a 3-category $\Psi_{\mathbb{L}}(\mathbb{G})$ which we can see as the 3-category of extensions of complexes of Picard \mathbf{S} -2-stacks of \mathbb{L} by \mathbb{G} , considering \mathbb{G} as a complex of length zero concentrated in degree 0. This 3-category is a generalization to Picard \mathbf{S} -2-stacks of the one introduced by Grothendieck in [9] for abelian sheaves.

Definition 5.1. Let $\Psi_{\mathbb{L}}(\mathbb{G})$ be the 3-category whose objects are pairs (\mathbb{E}, T) where $\mathbb{E} = (I : \mathbb{G} \rightarrow \mathbb{E}, \mathbb{E}, J : \mathbb{E} \rightarrow \mathbb{P}, \varepsilon)$ is an extension of \mathbb{P} by \mathbb{G} and $T = (T, \mu, \Upsilon)$ is a trivialization of the extension $(D^{\mathbb{Q}})^*\mathbb{E}$ of \mathbb{Q} by \mathbb{G} obtained as pull-back of \mathbb{E} via $D^{\mathbb{Q}} : \mathbb{Q} \rightarrow \mathbb{P}$. We require that the trivialization T is compatible with the complex \mathbb{L} , i.e. it satisfies the following conditions:

- (1) the trivialization $(D^{\mathbb{R}})^*T$ of $(D^{\mathbb{R}})^*(D^{\mathbb{Q}})^*\mathbb{E}$ is the trivialization arising from the equivalence of transitivity $(D^{\mathbb{R}})^*(D^{\mathbb{Q}})^*\mathbb{E} \cong (D^{\mathbb{Q}} \circ D^{\mathbb{R}})^*\mathbb{E}$ and from the morphism of additive 2-functors $\partial^{\mathbb{R}} : D^{\mathbb{Q}} \circ D^{\mathbb{R}} \Rightarrow 0$;
- (2) the morphism of additive 2-functor $(D^{\mathbb{S}})^*(D^{\mathbb{R}})^*T \Rightarrow 0$ arises from the 2-isomorphism of transitivity $(D^{\mathbb{S}})^*(D^{\mathbb{R}})^*T \cong (D^{\mathbb{R}} \circ D^{\mathbb{S}})^*T$ and from the morphism of additive 2-functors $\partial^{\mathbb{S}} : D^{\mathbb{R}} \circ D^{\mathbb{S}} \Rightarrow 0$;
- (3) the morphism of additive 2-functor $(D^{\mathbb{T}})^*(D^{\mathbb{S}})^*(D^{\mathbb{R}})^*T \Rightarrow 0$ is compatible with the modification of morphisms of additive 2-functors $\Delta_{(\mathbb{T}, \mathbb{S}, \mathbb{R})}$ (5.1) underlying the complex \mathbb{L} .

We left to the reader the description of the 1-, 2-, 3-arrows of the 3-category $\Psi_{\mathbb{L}}(\mathbb{G})$. To have the trivialization $T = (T, \mu, \Upsilon)$ [3, Def 8.3] is the same thing as to have a lifting of $D^{\mathbb{Q}}$ along J , i.e. an additive 2-functor $T : \mathbb{Q} \rightarrow \mathbb{E}$, a morphism of additive 2-functor $\mu : J \circ T \Rightarrow D^{\mathbb{Q}}$, and a modification of morphisms of additive 2-functors Υ involving T, μ and all additive 2-functors and morphisms of additive 2-functors underlying the extension \mathbb{E} and the trivial extension $\mathbb{Q} \times \mathbb{G}$. The compatibility between the trivialization T and the complex \mathbb{L} can be rewritten in the following way:

- (1) it exists a pair (ν, Φ) consisting of a morphism of additive 2-functor $\nu : T \circ D^{\mathbb{R}} \Rightarrow 0$ and a modification of morphisms of additive 2-functors

$$\begin{array}{ccccc}
(JT)D^{\mathbb{R}} & \xrightarrow{a} & J(TD^{\mathbb{R}}) & \xrightarrow{J*\nu} & J0 \\
\mu * D^{\mathbb{R}} \Downarrow & & \Downarrow \Phi & & \Downarrow \\
D^{\mathbb{Q}}D^{\mathbb{R}} & \xrightarrow{\partial^{\mathbb{R}}} & & \xrightarrow{\partial^{\mathbb{R}}} & 0
\end{array}$$

- (2) it exists a modification of morphisms of additive 2-functors

$$\begin{array}{ccccc}
 (TD^{\mathbb{R}})D^{\mathbb{S}} & \xrightarrow{a} & T(D^{\mathbb{R}}D^{\mathbb{S}}) & \xrightarrow{T*\partial^{\mathbb{S}}} & T0 \\
 \nu*D^{\mathbb{S}} \Downarrow & & \Downarrow_{\Theta} & & \Downarrow \\
 0D^{\mathbb{S}} & \xrightarrow{\quad\quad\quad} & & & 0
 \end{array}$$

- (3) the pasting of the modifications in the diagram

$$\begin{array}{ccccccc}
 & & (T(D^{\mathbb{R}}D^{\mathbb{S}}))D^{\mathbb{T}} & \xRightarrow{\quad} & T((D^{\mathbb{R}}D^{\mathbb{S}})D^{\mathbb{T}}) & \xRightarrow{\quad} & T(D^{\mathbb{R}}(D^{\mathbb{S}}D^{\mathbb{T}})) \\
 & \nearrow & \Downarrow & \Downarrow_{\circ(a,\partial^{\mathbb{S}})} & \Downarrow & & \Downarrow \\
 ((TD^{\mathbb{R}})D^{\mathbb{S}})D^{\mathbb{T}} & & (T0)D^{\mathbb{T}} & \xRightarrow{\quad} & T(0D^{\mathbb{T}}) & \xrightarrow{T*\Delta_{(D^{\mathbb{R}},D^{\mathbb{S}},D^{\mathbb{T}})}} & T(D^{\mathbb{R}}0) \\
 \Downarrow & \Downarrow_{\Theta*D^{\mathbb{T}}} & \Downarrow & \Downarrow_{\simeq} & \Downarrow & & \Downarrow \\
 (0D^{\mathbb{S}})D^{\mathbb{T}} & \xRightarrow{\quad\quad\quad} & 0D^{\mathbb{T}} & \xRightarrow{\quad\quad\quad} & 0 & \xRightarrow{\quad\quad\quad} & T0
 \end{array}$$

is equal to the pasting of the modifications in the diagram

$$\begin{array}{ccccccc}
 & & ((TD^{\mathbb{R}})D^{\mathbb{S}})D^{\mathbb{T}} & \xRightarrow{\quad} & (T(D^{\mathbb{R}}D^{\mathbb{S}}))D^{\mathbb{T}} & \xRightarrow{\quad} & T((D^{\mathbb{R}}D^{\mathbb{S}})D^{\mathbb{T}}) \\
 & \nwarrow & \Downarrow_{\circ(a,D^{\mathbb{R}})} & \nwarrow & \Downarrow_{\pi(T,D^{\mathbb{R}},D^{\mathbb{S}},D^{\mathbb{T}})} & & \Downarrow \\
 T((D^{\mathbb{R}}D^{\mathbb{S}})D^{\mathbb{T}}) & \xRightarrow{\quad} & T(D^{\mathbb{R}}(D^{\mathbb{S}}D^{\mathbb{T}})) & \xleftarrow{\quad} & (TD^{\mathbb{R}})(D^{\mathbb{S}}D^{\mathbb{T}}) & \xRightarrow{\quad} & T(D^{\mathbb{R}}(D^{\mathbb{S}}D^{\mathbb{T}})) \\
 \Downarrow & \Downarrow & \Downarrow & \Downarrow_{\circ(D^{\mathbb{R}},\partial^{\mathbb{T}})} & \Downarrow & \Downarrow_{\circ(a,\partial^{\mathbb{S}})} & \Downarrow \\
 T(0D^{\mathbb{T}}) & \xrightarrow{T*\Delta_{(D^{\mathbb{R}},D^{\mathbb{S}},D^{\mathbb{T}})}} & T(D^{\mathbb{R}}0) & \xleftarrow{\quad} & (TD^{\mathbb{R}})0 & \xRightarrow{\quad} & T(D^{\mathbb{R}}0) \\
 \Downarrow & \Downarrow & \Downarrow & \Downarrow_{\simeq} & \Downarrow & & \Downarrow \\
 0 & \xRightarrow{\quad\quad\quad} & & & & & T0
 \end{array}$$

Let \mathbb{P} be a Picard \mathbf{S} -2-stack and denote by $\mathbb{Z}[\mathbb{P}]$ the Picard \mathbf{S} -2-stack generated by it (Definition 3.4). For any object U of \mathbf{S} , if p_1, p_2, p_3 are objects of $\mathbb{P}(U)$, we denote by $[p_1, p_2]$, $[p_1, p_2, p_3]$ the objects of $\mathbb{Z}[\mathbb{P}^2](U)$ and $\mathbb{Z}[\mathbb{P}^3](U)$ respectively. Now we switch from cohomological to homological notation. Consider the following complex of Picard \mathbf{S} -2-stacks:

$$(5.2) \quad \mathbb{L}_*(\mathbb{P}) : \quad 0 \longrightarrow \mathbb{L}_4(\mathbb{P}) \xrightarrow{D_3} \mathbb{L}_3(\mathbb{P}) \xrightarrow{D_2} \mathbb{L}_2(\mathbb{P}) \xrightarrow{D_1} \mathbb{L}_1(\mathbb{P}) \xrightarrow{D_0} \mathbb{L}_0(\mathbb{P}) \longrightarrow 0$$

with $\mathbb{L}_0(\mathbb{P}) = \mathbb{Z}[\mathbb{P}]$, $\mathbb{L}_1(\mathbb{P}) = \mathbb{Z}[\mathbb{P}^2]$, $\mathbb{L}_2(\mathbb{P}) = \mathbb{Z}[\mathbb{P}^2] + \mathbb{Z}[\mathbb{P}^3]$, $\mathbb{L}_3(\mathbb{P}) = \mathbb{Z}[\mathbb{P}^4] + \mathbb{Z}[\mathbb{P}^3]$, and $\mathbb{L}_4(\mathbb{P}) = \mathbb{Z}[\mathbb{P}^5] + \mathbb{Z}[\mathbb{P}^4]$ in degrees 0,1,2,3 and 4 respectively, and with the differential operators

defined as follows:

$$\begin{aligned}
(5.3) \quad D_0[p_1, p_2] &= [p_1 + p_2] - [p_1] - [p_2]; \\
D_1[p_1, p_2] &= [p_1, p_2] - [p_2, p_1]; \\
D_1[p_1, p_2, p_3] &= [p_1 + p_2, p_3] - [p_1, p_2 + p_3] + [p_1, p_2] - [p_2, p_3]; \\
D_2[p_1, p_2, p_3, p_4] &= [p_1, p_2, p_3] + [p_1, p_2 + p_3, p_4] + [p_2, p_3, p_4] - [p_1 + p_2, p_3, p_4] \\
&\quad - [p_1, p_2, p_3 + p_4]; \\
D_2[p_1, p_2, p_3] &= [p_2, p_3, p_1] + [p_1, p_2 + p_3] + [p_1, p_2, p_3] - [p_1, p_3] - [p_2, p_1, p_3] - [p_1, p_2]; \\
D_3[p_1, p_2, p_3, p_4, p_5] &= [p_2, p_3, p_4, p_5] + [p_1, p_2 + p_3, p_4, p_5] + [p_1, p_2, p_3, p_4 + p_5] \\
&\quad - [p_1, p_2, p_3 + p_4, p_5] - [p_1, p_2, p_3, p_4] - [p_1 + p_2, p_3, p_4, p_5]; \\
D_3[p_1, p_2, p_3, p_4] &= [p_1, p_2, p_3, p_4] + [p_1, p_2, p_3 + p_4] + [p_1, p_3, p_4] + [p_2, p_1, p_3, p_4] \\
&\quad - [p_1, p_2 + p_3, p_4] - [p_2, p_3, p_4, p_1] - [p_2, p_3, p_1, p_4] - [p_1, p_2, p_3]
\end{aligned}$$

Let \mathbb{G} be a Picard **S**-2-stack.

Proposition 5.2. *The 3-category $\mathcal{E}xt(\mathbb{P}, \mathbb{G})$ of extensions of \mathbb{P} by \mathbb{G} is tri-equivalent to the 3-category $\Psi_{\mathbb{L}(\mathbb{P})}(\mathbb{G})$:*

$$\mathcal{E}xt(\mathbb{P}, \mathbb{G}) \cong \Psi_{\mathbb{L}(\mathbb{P})}(\mathbb{G}).$$

Proof. By Corollary 0.2, we know that:

- an extension of $\mathbb{Z}[\mathbb{P}]$ by \mathbb{G} is a $\mathbb{G}_{\mathbb{P}}$ -torsor,
- an extension of $\mathbb{Z}[\mathbb{P}^2]$ by \mathbb{G} is a $\mathbb{G}_{\mathbb{P}^2}$ -torsor,
- an extension of $\mathbb{Z}[\mathbb{P}^2] + \mathbb{Z}[\mathbb{P}^3]$ by \mathbb{G} consists of a couple of a $\mathbb{G}_{\mathbb{P}^2}$ -torsor and a $\mathbb{G}_{\mathbb{P}^3}$ -torsor,
- an extension of $\mathbb{Z}[\mathbb{P}^4] + \mathbb{Z}[\mathbb{P}^3]$ by \mathbb{G} consists of a couple of a $\mathbb{G}_{\mathbb{P}^4}$ -torsor and a $\mathbb{G}_{\mathbb{P}^3}$ -torsor,
- an extension of $\mathbb{Z}[\mathbb{P}^5] + \mathbb{Z}[\mathbb{P}^4]$ by \mathbb{G} consists of a couple of a $\mathbb{G}_{\mathbb{P}^5}$ -torsor and a $\mathbb{G}_{\mathbb{P}^4}$ -torsor.

Therefore an object (\mathbb{E}, T) of $\Psi_{\mathbb{L}(\mathbb{P})}(\mathbb{G})$ consists of an extension \mathbb{E} of $\mathbb{Z}[\mathbb{P}]$ by \mathbb{G} , i.e. a $\mathbb{G}_{\mathbb{P}}$ -torsor \mathbb{E} , and a trivialization T of the extension $D_0^*\mathbb{E}$ of $\mathbb{Z}[\mathbb{P}^2]$ by \mathbb{G} obtained as pull-back of \mathbb{E} via $D_0 : \mathbb{Z}[\mathbb{P}^2] \rightarrow \mathbb{Z}[\mathbb{P}]$, i.e. a trivialization T of the $\mathbb{G}_{\mathbb{P}^2}$ -torsor $D_0^*\mathbb{E}$. This trivialization can be interpreted as a morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors $M : p_1^* \mathbb{E} \wedge p_2^* \mathbb{E} \rightarrow \otimes^* \mathbb{E}$, where $p_i : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ is the i -th projection of $\mathbb{P} \times \mathbb{P}$ on \mathbb{P} and $\otimes : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ is the group law of \mathbb{P} .

Concerning the compatibility between the trivialization T and the complex $\mathbb{L}(\mathbb{P})$, we have:

- (1) the compatibility of T with the morphism of additive 2-functors $D_0 \circ D_1 \Rightarrow 0$ imposes on the datum (\mathbb{E}, M) two relations through the two torsors over \mathbb{P}^2 and \mathbb{P}^3 : these two relations are the 2-morphism of $\mathbb{G}_{\mathbb{P}^3}$ -torsors α described in Proposition 4.1 (3) and the 2-morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors χ described in Proposition 4.1 (5);
- (2) the compatibility between the two morphisms of additive 2-functors $D_2^* D_1^* T \Rightarrow 0$ and $D_1 \circ D_2 \Rightarrow 0$ imposes on the 2-morphisms of torsors α and χ two relations through the two torsors over \mathbb{P}^4 and \mathbb{P}^3 : these two relations are the 3-morphism of $\mathbb{G}_{\mathbb{P}^4}$ -torsors \mathfrak{a} described in Proposition 4.1 (4) and the 3-morphism of $\mathbb{G}_{\mathbb{P}^3}$ -torsors \mathfrak{c}_1 described in Proposition 4.1 (7);
- (3) the compatibility between the morphism of additive 2-functors $D_3^* D_2^* D_1^* T \Rightarrow 0$ and the modification $\Delta_{(D_1, D_2, D_3)}$ underlying the complex (5.2) imposes on the 3-morphisms of torsors \mathfrak{a} and \mathfrak{c}_1 two relations through the two torsors over \mathbb{P}^5 and \mathbb{P}^4 : these two relations are the equalities (4.1) and (4.2).

Since we are dealing with extensions of Picard **S**-2-stacks, there exists a 3-morphism of $\mathbb{G}_{\mathbb{P}^2}$ -torsors $\mathfrak{s} : \chi \circ \chi \Rightarrow \text{id}$ and a 3-morphism of $\mathbb{G}_{\mathbb{P}}$ -torsors $\mathfrak{p} : D^* \chi \Rightarrow \text{id}$ (here $D : \mathbb{P} \rightarrow \mathbb{P} \times \mathbb{P}$ is the

diagonal morphism of \mathbf{S} -2-stacks). Hence by Theorem 4.1 the object $(\mathbb{E}, M, \alpha, \mathbf{a}, \chi, \mathbf{s}, \mathbf{c}_1, \mathbf{c}_2, \mathbf{p})$ of $\Psi_{\mathbb{L}(\mathbb{P})}(\mathbb{G})$ is an extension of \mathbb{P} by \mathbb{G} . The remaining detail are left to the reader. \square

Remark 5.3. (1) The complex $\mathbb{L}(\mathbb{P})$ (5.2) furnishes the data underlying the Picard \mathbf{S} -2-stack \mathbb{E} which do not involve inverse or weak inverse.

(2) Generalizing to Picard \mathbf{S} -2-stacks the proof of [2, Thm 8.2] we have that $\pi_{-i+1}(\Psi_{\mathbb{L}(\mathbb{P})}(\mathbb{G})) = \text{Hom}_{\mathcal{D}(\mathbf{S})}([\mathbb{P}]^{bb}, [\mathbb{G}]^{bb}[i])$ for $i = 1, 0, -1, -2$. Together with Proposition 5.2, the above equality furnishes another proof of [3, Thm 1.1].

Let $\mathbb{L}^\bullet : 0 \rightarrow \mathbb{T} \rightarrow \mathbb{S} \rightarrow \mathbb{R} \rightarrow \mathbb{Q} \rightarrow \mathbb{P} \rightarrow 0$ and $\mathbb{L}'^\bullet : 0 \rightarrow \mathbb{T}' \rightarrow \mathbb{S}' \rightarrow \mathbb{R}' \rightarrow \mathbb{Q}' \rightarrow \mathbb{P}' \rightarrow 0$ be two complexes of Picard \mathbf{S} -2-stacks with \mathbb{P}, \mathbb{P}' in degree 0, \mathbb{Q}, \mathbb{Q}' in degree -1, \mathbb{R}, \mathbb{R}' in degree -2, \mathbb{S}, \mathbb{S}' in degree -3, and \mathbb{T}, \mathbb{T}' in degree -4. For any Picard \mathbf{S} -2-stack \mathbb{G} , a morphism $F^\bullet = (F^{-4}, F^{-3}, F^{-2}, F^{-1}, F^0) : \mathbb{L}'^\bullet \rightarrow \mathbb{L}^\bullet$ of complexes of Picard \mathbf{S} -2-stacks induces a canonical 3-functor

$$(F^\bullet)^* : \Psi_{\mathbb{L}^\bullet}(\mathbb{G}) \rightarrow \Psi_{\mathbb{L}'^\bullet}(\mathbb{G}).$$

In fact, if (\mathbb{E}, T) is an object of $\Psi_{\mathbb{L}^\bullet}(\mathbb{G})$, $(F^\bullet)^*(\mathbb{E}, T)$ is the object (\mathbb{E}', T') with \mathbb{E}' the extension $(F^0)^*\mathbb{E}$ of \mathbb{P}' by \mathbb{G} obtained as pull-back of \mathbb{E} via $F^0 : \mathbb{P}' \rightarrow \mathbb{P}$, and T' the trivialization $(F^{-1})^*T$ of $(D^{\mathbb{Q}'})^*\mathbb{E}'$ induced by the trivialization T of $(D^{\mathbb{Q}})^*\mathbb{E}$. Generalizing [2, Prop 8.5] to Picard \mathbf{S} -2-stacks, we have that $(F^\bullet)^* : \Psi_{\mathbb{L}^\bullet}(\mathbb{G}) \rightarrow \Psi_{\mathbb{L}'^\bullet}(\mathbb{G})$ is a tri-equivalence of 3-categories if and only if

$$H^i(\text{Tot}(F^\bullet)) : H^i(\text{Tot}([\mathbb{L}'^\bullet]^{bb})) \rightarrow H^i(\text{Tot}([\mathbb{L}^\bullet]^{bb}))$$

is an isomorphism for any i . We apply this fact to the morphism of complexes $\epsilon : \mathbb{L}(\mathbb{P}) \rightarrow \mathbb{P}$ defined by the additive 2-functor $\epsilon : \mathbb{L}_0(\mathbb{P}) = \mathbb{Z}[\mathbb{P}] \rightarrow \mathbb{P}, \epsilon([p]) = p$, for any $p \in \mathbb{P}$ (here we consider \mathbb{P} as a length 0 complex concentrated in degree 0). Since by definition $\Psi_{\mathbb{P}}(\mathbb{G})$ is tri-equivalent to $\mathcal{E}\text{xt}(\mathbb{P}, \mathbb{G})$, Proposition 5.2 implies that the 3-functor $(\epsilon)^* : \Psi_{\mathbb{P}}(\mathbb{G}) \rightarrow \Psi_{\mathbb{L}(\mathbb{P})}(\mathbb{G})$ is a tri-equivalence and so $H_i(\text{Tot}(\epsilon))$ is an isomorphism for any i :

Corollary 5.4. *The complex $\mathbb{L}(\mathbb{P})$ (5.2) is a canonical flat partial resolution of the Picard \mathbf{S} -2-stack \mathbb{P} .*

Hence, the following complex of Picard \mathbf{S} -2-stacks is exact

$$0 \longrightarrow \mathbb{Z}[\mathbb{P}^5] + \mathbb{Z}[\mathbb{P}^4] \xrightarrow{D_3} \mathbb{Z}[\mathbb{P}^4] + \mathbb{Z}[\mathbb{P}^3] \xrightarrow{D_2} \mathbb{Z}[\mathbb{P}^2] + \mathbb{Z}[\mathbb{P}^3] \xrightarrow{D_1} \mathbb{Z}[\mathbb{P}^2] \xrightarrow{D_0} \mathbb{Z}[\mathbb{P}] \xrightarrow{\epsilon} \mathbb{P} \longrightarrow 0$$

Applying to it the left exact contra-variant 3-functor $\mathcal{E}\text{xt}(-, \mathbb{G})$, we get

$$\begin{aligned} 0 \rightarrow \mathcal{E}\text{xt}(\mathbb{P}, \mathbb{G}) \xrightarrow{(\epsilon)^*} \mathcal{E}\text{xt}(\mathbb{Z}[\mathbb{P}], \mathbb{G}) \xrightarrow{D_0^*} \mathcal{E}\text{xt}(\mathbb{Z}[\mathbb{P}^2], \mathbb{G}) \xrightarrow{D_1^*} \mathcal{E}\text{xt}(\mathbb{Z}[\mathbb{P}^2] + \mathbb{Z}[\mathbb{P}^3], \mathbb{G}) \xrightarrow{D_2^*} \dots \\ \dots \xrightarrow{D_3^*} \mathcal{E}\text{xt}(\mathbb{Z}[\mathbb{P}^4] + \mathbb{Z}[\mathbb{P}^3], \mathbb{G}) \xrightarrow{D_4^*} \mathcal{E}\text{xt}(\mathbb{Z}[\mathbb{P}^5] + \mathbb{Z}[\mathbb{P}^4], \mathbb{G}) \end{aligned}$$

Finally, using Corollary 0.2 we obtain Corollary 0.4.

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